Independent Study

MATH-SHU 997

Professor Roberto Fernandez

Binxi Li

Catalogue

Chapter 2 Brownian Motion

1.Brownian Motion

Def 3.3.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space. For each $w \in \Omega$, suppose there is a continous function $W(t)$ for $t \geqslant 0$ that satisfies $W(0) = 0$ and that depends on W. Then $W(t)$, to, is a Brownian motion if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments
$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \cdots W(t_m) - W(t_{m-1})$

are independent and each of these increments is normally distributed with

$$\begin{aligned}
&E[W(t_{i+1}) - W(t_i)] = 0 \\
&\operatorname{Var}[W(t_{i+1}) - W(ti)] = t_{i+1} - t_i \\
&0 \leq s \leq t \\
&E[W(s)W(t)] = E\big[w(s)(w(t) - w(s)) + w^2(s)\big] \\
&\ = E[w(s)] \cdot E[W(\tau) - w(s)] + E\big[w^2(s)\big] \\
&\ = 0 + \operatorname{Var}[w(s)] = s
\end{aligned}$$

moment-generating function for Brownian motion (ie, for the m-dimensiond random vector $(w(t_1), w(t_2), \cdots w(t_m))$ is

$$\begin{aligned}
&\ \varphi(u_1, u_2, \cdots u_n) \\
=&\mathrm{Ee}\{u_m w(t_m) + u_{m-1} w(t_{m-1}) + \cdots + u_1 w(t_1)\} \\
=&\exp\left\{\frac{1}{2}(u_1 + u_2 + \cdots + u_m)^2 t_1 + \frac{1}{2}(u_2 + u_3 + \cdots + u_m)^2 (t_2 - t_1)\right. \\
&\left. + \cdots + \frac{1}{2}(u_{m-1} + u_m)^2 (t_{m-1} - t_{m-2}) + \frac{1}{2} u_m^2 (t_m - t_{m-1})\right\}
\end{aligned}$$

2.Filtration of Brownian Motion

Def: A filtration for the Brownian motion is a collection of σ-algebras $\mathcal{F}(t), t \geqslant 0$ satisfying

(i) information accumulates

(ii) adaptivity

(iii) Independents of future increaments

For $0 \leq t < u$, the increament $w(u) - w(t)$ is independent of $\mathcal{F}(t)$.

$\Delta(t), t \geqslant 0$, be stochastic process, we say that $\Delta(t)$ is adapted to the filtration $\mathcal{F}(t)$ if for each $t \geqslant 0$ the random variable $\Delta(t)$ is $\mathcal{F}(t)$-measurable

2 possibilities for the filtration $F(t)$

(1) Let $\mathcal{F}(t)$ contain only the information obtained by observing the Brownian motion itself up to time t

(2) include in $\mathcal{F}(t)$ information obtained by observing the Brownian motion and one or more process

3.Quadratic Variation

First- Order Variation

$$
\begin{aligned}
FV_T(f) &= [f(t_1) - f(0)] + [f(t_2) - f(t_1)] + [f(T) - f(t_2)] \\
&= \int_0^{t_1} f'(t)dt + \int_{t_1}^{t_2} \big(-f'(t)\big)dt + \int_{t_2}^{T} f'(t)dt \\
&= \int_0^{T} |f'(t)|dt \\
F_{V_T(f)} &= \lim_{\|\pi_i\| \to \|0} \sum_{j=0}^{n-1} |f(t+j) - f(tj)|
\end{aligned}
$$

Def 3.4.1

Let $f(t)$ be a function defined for $0 \leqslant t \leqslant T$. The quadratic variation of f up to time T is $[f,f](T) = \lim_{\|\pi_i\| \to |0} \sum_{j=0}^{n-1} |f(t+j) - f(tj)|$

where $\pi = \{t_0, t_1, \cdots, t_n\}$. and $0 < t_0 < t_1 < \cdots < t_n = T$

RMK 3.4.2 suppose the function f has a continous derivatives. Then

$$
\sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2 = \sum_{j=0}^{n-1} \mid f'_{t_j|^2} \mid \Bigg((t_{j+1} - t_j)^2 \leq ||\pi \Bigg| \cdot \sum_{j=0}^{n-1} \Bigg| f'(t_j) \Bigg| \Bigg|^2 |t_{j+1} - t_j|
$$

thus $[f,f](T) \leq \lim_{\|\pi\| \to 0} \left[||\pi\Big|| \cdot \sum_{i=0}^{n-1} \Big| f\Big(t_j\Big)^2 (t_{j-1} - t_j) \right]$

$$
= \lim_{\|\pi\| \to 0} \|\pi\| \cdot \int_0^T |f'(t)\Big|^2 dt = 0
$$

Most functions have continous derivatives, and hence their quadratic variations are 0.

Thm 3.4.3. Let W be a Brownian motion. Then $[W,W](T) = T$ for all $T \geqslant 0$ almost surely

sampled quadratic variations

$$
\begin{aligned}
&Q_\pi = \sum_{j=0}^{h-1} (w(t_{j+1}) - w(t_j))^2 \\
&EQ\pi = \sum_{j=0}^{n-1} E\Big[\big(w_{(j+1)}\big) - w(t_j)\big)^2\Big] = \sum_{j=0}^{n-1} (t_{j+1} - t_j) = T \\
&\mathrm{Var}(Q\pi) = \sum_{j=0}^{n-1} \mathrm{Var}\Big[(w(t_{j+1}) - w(t_j))^2\Big] - \sum_{j=0}^{n-1} 2(t_{j-1} - t_j)^2 = \sum_{j=0}^{n-1} 21 \mid -\|(i_{j+1} - t_j) \\
&= 2\|\pi\|T
\end{aligned}
$$

RMK 3.4.4

$$
\begin{aligned}
E[w(t_{j+1}) - w(t_j))^2\Big] &= t_{j+1} - t_j \\
\mathrm{Var}\Big[(w(t_{j+1}) - w(t_j))^2\Big] &= 2(t_{j+1} - t_j)^2
\end{aligned}
$$

we therefore claim that $(W(t_{j+1}) - W(t_j))^2 \approx t_{j+1} - t_j$

standard normal random variable

$$Y_{j+1}=\frac{w(t_{j+1})-w(t_j)}{\sqrt{t_{j+1}-t_j}}\quad t_{j+1}-t_j=\frac{T}{n}$$

$$(w(t_{j+1})-w(t_j))^2=T\cdot\frac{Y_{j+1}^2}{n}$$

The quadratic variation of Brownian motion is the source of volating in asset prices driven by Brownian motion. We shall eventually scale Bromnican notion, sometimes in time and path - dependert ways, in order to vary the rate at which volatility enters the asset prices.

RMK. 3.4.5 Let $\pi=t_0,t_1\cdots$ be a partition of $[0,T]$. In addition to computing the quadratic variation of Brownian motion

$$\lim_{\|\pi\|\to 0}\sum_{j=0}^{k-1}(w(t_{j+1})-w(t_j))^2=T$$

4.Volatility of Geometric Brownian motion

Let α and $\sigma>0$ be constants and define the geometric Brownian notion
$S(t)=S(0)\exp\{\sigma W(t)+\left(\alpha-\frac{1}{2}\sigma^2\right)t\}$

asset - price model used in the Black - Scholes Merton option pricing formula

$$\log\frac{S(t_{j+1})}{S(t_j)}=\sigma(w(t_{j+1})-w(t_j))+\left(\alpha-\frac{1}{2}\sigma^2\right)(t_{j+1}-t_j)$$

$$\sum_{j=0}^{m-1}\left(\log\frac{S(t_{j+1})}{S(t_j)}\right)^2$$

$$=\sigma^2\sum_{j=0}^{m-1}(w(t_{j+1})-w(t_j))^2+\left(\alpha-\frac{1}{2}\sigma^2\right)\sum_{j=0}^{m-1}(t_{j+1}-t_j)^2$$

$$+2\sigma\left(\alpha-\frac{1}{2}\sigma^2\right)\sum_{j=0}^{m-1}(w(t_{j+1})-w(t_j))(t_{j+1}-t_j)$$

$$\|\pi\|=\max\,(t_{j+1}-t_j)$$

When $\|T\|$ is small, all be cone $\sigma^2(T_2-T_1)$

$$\frac{1}{T_2-T_1}\sum_{j=0}^{m-1}\left(\log\frac{S(t_{j+1})}{S(t_j)}\right)^2\approx\sigma^2$$

5.Markov Property

Thm. Let $W(t),t\geqslant 0$ be a Brownian motion and let $\mathcal{F}(t),t\geqslant 0$ be a filtration for this Brownian motion. Then $W(t),t\geqslant 0$, is a Markov process.

$$E[f(w(t)) \mid f(s)] = g(w(s))$$

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(w+x) e^{-\frac{\omega^2}{2(t-s)}} dw$$

$$T = t - s \quad y = \omega + x$$

$$g(x) = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(y-x)^2}{2T}} dy$$

$$\text{transition density } P(T, x, y) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{(y-x)^2}{2T}}$$

$$g(x) = \int_{-\infty}^{\infty} f(y) P(T, x, y) dy$$

$$E\Big[f(w(t) \mid f(s)] = \int_{-\infty}^{\infty} f(y) P(T, w(s), y) dy$$

1. σ-algebra

Def: A family $\mathcal{F}$ of subsets of a sample space σ is a σ-algebra if it satisfies following properties

(S1) $\mathcal{F}$ contains the external sets, The empty set ϕ and the universe Ω belong to F

(S2) $\mathcal{F}$ is closed under complements; If a set of outcomes A belong to $\mathcal{F}$ then so does its complement $A^c := \Omega \setminus A$

(S3) $\mathcal{F}$ is closed under countable unions: For any finite or countably infinite famliy of sets $A_i \in \mathcal{F}$, its union also belong to $\mathcal{F}$

We consider σ-algebras $\mathcal{F}_n$ representing market information up to the nth period.

Further properties of σ-algebras

(1) σ algebras are closed under set difference : $A, B \in F \;\; A \setminus B \in \mathcal{F}$

(2) σ algepras are closed under countable intersections: If I is finite or countable, $A_i \in \mathcal{F}, i \in I \Rightarrow \bigcap_{i \in I} A_i \in \mathcal{F}$

2. Filtration

- deterministic process.

A deterministic process has each value exactly determined once the past is known:

$$X_{n+1} = H_n(X_0, X_1, \ldots, X_n)$$

- stochastic process:

the past only determines the probability distribution of the next value, $P(a \leq X_{n+1} \leq b \mid \mathcal{F})$

Def: A filtration in a measurable spare $(\Omega, \mathcal{F})$ is a sequence of nested σ -algebras

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \ldots \subset \mathcal{F}$$

adapted:

A stochastic process $\underline{X} = (X_n)_{n \geqslant 0}$ is adapted to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n \geqslant 0}$ if X_n is $\mathcal{F}_n$ - measurable for each n.

3. Martingale

Def:
A stochastic process $\underline{M} = (M_n)_{n \geq 0}$ adapted to a filtration $\underline{F} = (\mathcal{F}_n)_{n \geqslant 0}$ is a

martingale if $E(M_{n+1} \mid F_n) = M_n$
sub-martingale if $E(M_{n+1} \mid F_n) \geqslant M_n \quad \rightarrow$ player super-martingale if
supermartingale if $E(M_{n+1} \mid F_n) \leq M_n \rightarrow$ casino

Corollary:
Let (M_r) be a process adapted to a certain filtration $\mathcal{F}$.
If (M_n) is a martingale, then $E(M_n) = E(M_k)$
If (M_n) is a submartingale, then $E(M_n) \geq E(M_k)$

If (M_n) is a supermartingale, then $E(M_n) \leq E(M_k)$
for all n $\geq$ k

setting $k = 0$, this corollary implies

$$E(M_n) \underset{\leq}{\overset{=}{\geqslant}} M_0 \quad \text{for all } n \geqslant 0$$

as M_0 is $\mathcal{F}_o$ measurable it is constant. $E(M_0) = M_0$

4.Markov processes

Def: A process $\underline{X}$ adapted to a filtration $\mathcal{F}$ is Markovian if for every function f and every n there wrists a function $g_{n.f}$ s.t, $E(f(X_{n+1}) \mid \mathcal{F}_n) = g_{n},_f\,(X_n)$

5. Geometric Brownian notion

A geometric Brownian notion is a continuous - time stochastic process in which the logarithm of the randomly varing quantity follows a Brownian notion

Def: suppose that $Z = \{Z_t, t \in [0, \infty)$ is standard Brownian motion and that $\mu \in \mathbb{R}$ and $\sigma \in (0, +\infty)$. Let $X_t = \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t\right], t \in [0, \infty)$

The stochastic process $x = \{X_t : t \in [0, \infty)\}$ is geometric Brownian motion with drift μ and volatility σ.

Note that the stochastic process

$\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t : t \in [0, \infty)\right\}$. is Brownian motion with drift parameter $\left(\mu - \frac{\sigma^2}{2}\right)$ and scalar parameter σ , so geometric Brownian motion is just exponential of this process.

$X_0 = 1$, the process starts at 1.

For $x_0 \in (0, +\infty)$, the process $\{_{x_0}X_t : t \in (0, +\infty)\}$ is geometric Brownian motion starting at x_0.

Geometric Brownian motion $x = \{X_t : t \in [0, \infty)\}$ satisfies the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dZ_t$$

Distributions

For $t \in (0, \infty), X_t$ has the lognornal distribution with parameters $\left(\mu - \frac{\sigma^2}{2}\right)t$ and $\sigma\sqrt{t}$. The probabilits density funceim f_t is given by

$$f_t(x) = \frac{1}{\sqrt{2\pi t}\sigma x} \exp\left(-\frac{\left[\ln(x) - \left(\mu - \frac{\sigma^2}{2}\right)t\right]^2}{2\sigma^2 t}\right), x \in (0, \infty)$$

For $t \in (0, \infty)$, the distribution function F_t of X_t is given by

$$F_t(x) = \Phi[\frac{\ln(x) - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}], x \in (0, \infty)$$

standard normal distribution

6.Moments

For $n \in \mathbb{N}$ ard $t \in [0, \infty]$,

$$E(X_t^n) = \exp\left[n\mu + \frac{\sigma^2}{2}\left(n^2 - n\right)t\right]$$

For $t \in [0, \infty)$

$$E(X_t) = e^{\mu t}$$

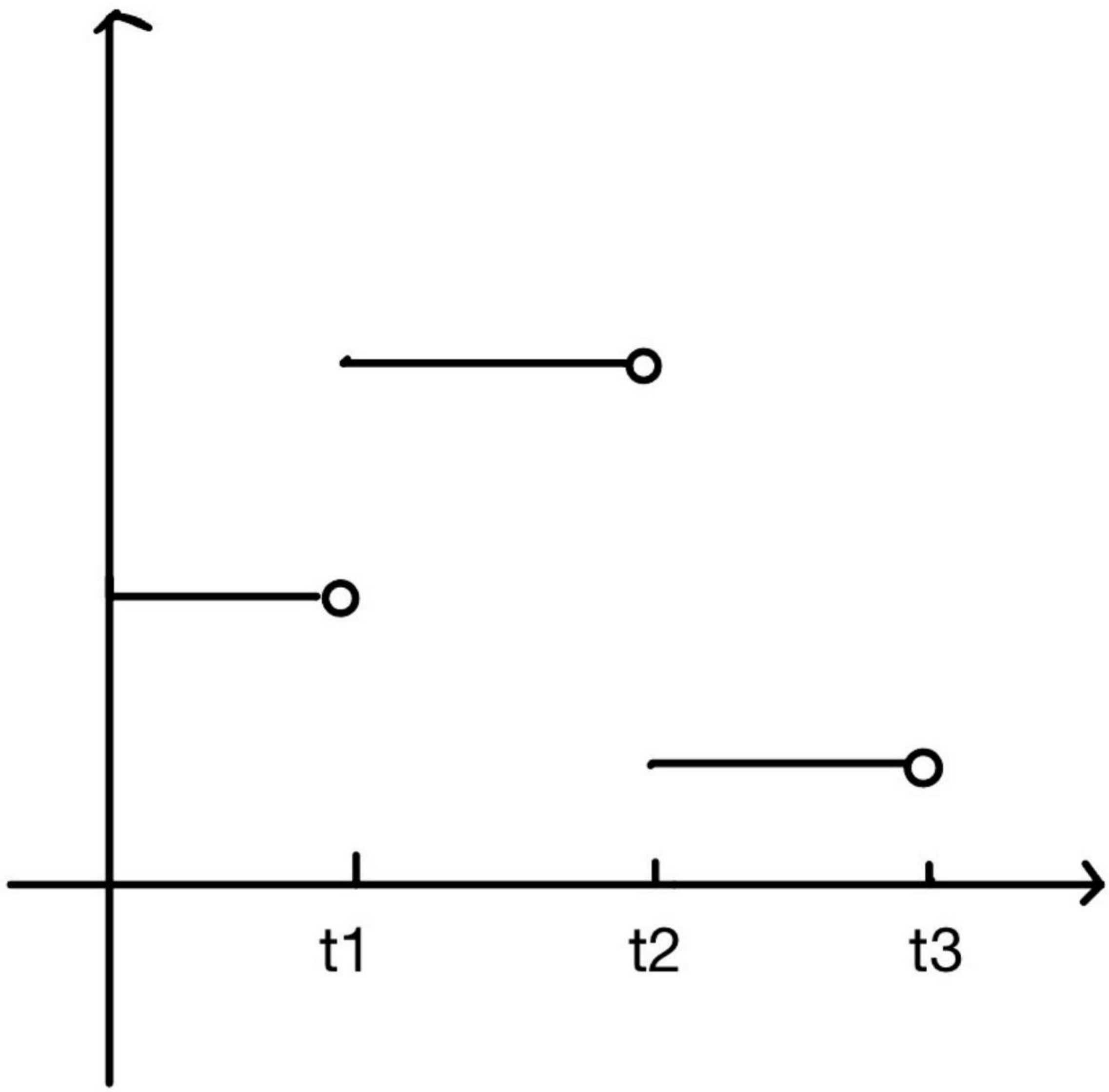
$$\mathrm{var}(X_t) = e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$$

1. Ito's Integral for Simple Integrands

$$\int_0^T \Delta(t)dW(t)$$

$\Delta(t)$: a simple process, constant on each subinterval $[t_j, t_{j+1})$

$\Delta(t)$



$I(t) = \Delta(t_0)[W(t) - W(t_0)] = \Delta(0)W(t)$
$I(t) = \Delta(0)W(t_1) + \Delta(t_1) \cdot [W(t) - W(t_1)] \quad t_1 \le t \le t_2$
$I(t) = \Delta(0)W(t_1) + \Delta(t_1) \cdot [W(t_2) - W(t_1)] + \Delta(t_2)][W(t) - W(t_2)\}$
$t_2 \le t \le t_3$

$$\text{if } t_k \leqslant t \leqslant t_{k+1}$$
$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)] \tag{4.2.2}$$
$$I(t) = \int_0^t \Delta(u) dW(u)$$

Thm 4.2.1 The *Itô* integral defined by (4.2.2) is a martingale

$$I(0) = 0$$
$$EI(t) = 0 \text{ for all } t \geqslant 0$$
$$\operatorname{Var} I(t) = EI^2(t)$$

The 4.2.2 The Ito Integral satisfies

$$EI^2(t) = E \int_0^t \Delta^2(u) du$$

Thm 4.2.3. The quadratic variation accumulated vp to time t by the 2 to integral is

$$[\mathrm{I}, \mathrm{I}]\ (t) = \int_0^t \Delta^2(u) du$$

RMK 4.2.4 The notations $I(t) = \int_0^t \Delta(u) dW(u)$

$$dI(t) = \Delta(t) dw(t)$$
$$I(t) = I(0) + \int_0^t \Delta(u) dW(u)$$

2. *Itô* - Doeblin Formula

$$\frac{d}{dt} f(W(t)) = f'(W(t)) W'(t)$$
$$df(w(t)) = f'(w(t)) w'(t) dt = f'(w(t)) dw(t)$$
$$df(w(t)) = f'(w(t)) dw(t) + \frac{1}{2} f''(w(t)) dt$$

This is the Itô-Doeblin formula in differential form

$$f\Big(w(t) - f(w(0)) = \int_0^t f'(w(u)) dw(u) + \frac{1}{2} \int_0^t f''(w(u)) du$$

Thm 4.4.1 (***Ito***- Doeblin formula for Brownian notion) Let $f(t, x)$ be a fer for which the partial cteivatives $f_t(t, x)$, $f_x(t, x)$, and $f_{xx}(t, x)$ are defined and continous, and let $W(t)$ be a Brownian motion. Then, for every $T \geqslant 0$

$$f(T,w(T)) = f(0,w(0)) + \int_0^T f_t(t,w(t))dt$$
$$+ \int_0^T f_x(t,w(t))dw(t) + \frac{1}{2}\int_0^T f_{xx}(t,w(t))dt$$

Taylor's formula

$$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1}-x_j) + \frac{1}{2}f''(x_j)(x_{j+1}-x_j)^2$$
$$f(W(T)) - f(W(0)) = \sum_{j=0}^{n-1}[f(W_{j+1}) - f(w(tj))]$$
$$= \sum_{j=0}^{n-1} f'(W(t_j))[W(t_{j+1}) - W(t_j)]$$
$$+ \frac{1}{2}\sum_{j=0}^{n-1} f''(W(t_j))[W(t_{j+1}) - W(t_j)]^2$$
$$f(x) = \frac{1}{2}x^2$$
$$RHS = \sum_{j=0}^{n-1} W(t_j)[W(t_{j+1}) - W(_j)] + \frac{1}{2}\sum_{i=0}^{n-1}[W(t_{j+1}) - W(t_j)]$$

$$f(W(T)) - f(W(0))$$
$$= \lim_{\|Tv\|\to 0}\sum_{j=0}^{n-1} W(t_j)[W(t_{j+1}) - W(t_j)] + \lim_{\|\pi\|\to 0}\frac{1}{2}\sum_{j=0}^{n-1}[W(t_{j+1}) - W(t_j)]^2$$
$$= \int_0^T w(t)dw(t) + \frac{1}{2}T$$
$$= \int_0^T f'(w(t))dw(t) + \frac{1}{2}\int_0^T f''(w(t))dt$$

If we take a function $f(t,x)$ of both the time variable t and the variable x, then

$$f(t_{j+1},x_{j+1}) - f(t_j,x_j)$$
$$= f_v(t_j,x_j)(t_{j+1}-t_j) + f_x(t_j,x_j)(x_{j+1}-x_j)$$
$$+ \frac{1}{2}f_{xx}(t_j,x_j)(x_{j+1}-x_j)^2 + f_{tx}(t_j,x_j)(t_{j+1}-t_j)(x_{j+1}-x_j)$$
$$+ \frac{1}{2}f_{tt}(t_j,x_j)(t_{j+1}-t_j)^2 + \text{ higher order terms}$$

replace x_j by $w(tj)$, replace x_{j+1} by $w(t_{j+1})$

$$f(T,W(T)) - f(0,W(0))$$
$$= \sum_{j=0}^{n-1}[f(t_{j+1},w(t_{j+1})) - f(t_j,w(t_j))]$$
$$= \sum_{j=0}^{n-1} f_t(t_j,w(t_j))(t_{j+1}-t_j) + \sum_{j=0}^{n-1} f_x(t_j,w(t_j))(w(t_{j+1}) - w(t_j))$$
$$+ \frac{1}{2}\sum_{j=0}^{n-1} f_{xx}(t_j,w(t_j))(w(t_j+1) - w(t_j))^2$$

$$+\sum_{j=0}^{n-1} f_{tx}(t_j, w(t_j))(t_{j+1}-t_j)(w(t_{j+1})-w(t_j))$$
$$+\sum_{2}^{n-1}\sum_{j=0}^{n-1} f_{tt}(t_j, w(t_j))(t_{j+1}-t_j)^2 + \text{ higher order terms}$$

Take the lim $\|\pi\| \to 0$, the left hand side is unaffected

$$\begin{aligned}
&\lim_{\|\pi|\to 0}\sum_{j=0}^{n-1} f_t(t_j, w(t_j))(t_{j+1}-t_j) = \int_0^T f_t(t, w(t))dt \\
&Ex: \frac{1}{2}W^2(T) = f(W(T)) - f(W(0)) \\
&= \int_0^T f'(w(t))dw(t) + \frac{1}{2}\int_0^t f''(w(t))dt \\
&= \int_0^T W(t)dW(t) + \frac{1}{2}T
\end{aligned}$$

3. Formula for *Itô* Process

Almost all stochastic processes, except those that have jumps, are Ito processes.

Def: Let $W(t), t \geqslant 0$, be a Brownian motion, and let $\mathcal{F}, t \geqslant 0$ be an associated filtration, An Ito process is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \theta(u)du$$

$X(0)$ is nonrandom, $\Delta(u), \theta(n)$ are adapted stochastic processes

Lemma 4.4.4 The quadratic variation of the ITo process

$$\begin{aligned}
[X,X](t) &= \int_0^t \Delta^2(u)du \\
dX(t) &= \Delta(t)dW(t) + \theta(t)dt \\
\text{is } \ dX(t)dX(t) &= \Delta^2(t)dw(t)dw(t) + 2\Delta(t)\theta(t)dw(t)dt \\
&+ \theta^2(t)dtdt \\
&= \Delta^2(t)dt \\
d(t)dW(t):\ dw(t)dt &= 0 \quad dtdt = 0
\end{aligned}$$

the total quadrate variation accumulated on $[0,T]$ is $\Big[X,X](t) = \int_0^t \Delta^2(u)du$

integral with respect to time; $R(t) = \int_0^t \theta(u)du$ integral with respect to Brownian motion; $I(t) = \int_0^t \Delta(u)dw(u)$ integral with respect to It rouses; integrals of the form $\int_0^t T(u)dX(u)$

$- \to$ the total quadratic variation accumulated on $[0,t]$

$$\text{is } [X,X](t) = \int_0^t \Delta^2(u)du$$

integral with respect to time: $R(t) = \int_0^t \theta(u)du$ integral with respect to Brownian motion; $I(t) = \int_0^t \Delta(u)dw(u)$ integral with respect to It processes; integrals of the form $\int_0^t T(n)dX(n)$

T is some adapted process

Definition 4.4.5 Let $X(t), t \geqslant 0$ be an $It\hat{o}$ process as described in def4.4.3. and let $T(t), t \geqslant 0$ be an adapted process. We define the integral with respect to an $It\hat{o}$ process

$$\int_0^t T(u)dx(u) = \int_0^t T(u)\Delta(u)dW(u) + \int_0^T T(u)\theta(u)du$$

(replace W(T) with $X(T)$)

$$f(T, X(T)) - f(0, X(0))$$
$$= \sum_{j=0}^{n-1} f_t\big(t_j, x_{(t_j)}\big)(t_{j+1} - t_j) + \sum_{j=0}^{n-1} f_x(t_j, x(t_j))(x(t_{j+1}) - x(t_j))$$
$$+ \frac{1}{2}\sum_{j=1}^{n-1} f_{xx}(t_j, x(t_j))(x(t_{+1}) - x(t_j))^2$$
$$+ \sum_{j=0}^{n-1} f_{tx}(t_j, x(t_j))(t_{j+1} - t_{j-1})(x(t_{j+1}) - x(t_j))$$
$$+ \frac{1}{2}\sum_{j=0}^{n-1} f_{tt}(t_j, X(t_j))(t_{j+1} - t_j)^2 + \text{ higher order terms}$$

The 4.4.6 Itô-Doeblin formula for an $It\hat{o}$ process

$X(t) = Ito\ process$

$f_t(t,x), f_x(t,x), f_{xx}(t,x)$ defined and continous

$$f(T, x(T))$$
$$\begin{aligned} =& f(0, X(0)) + \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))dX(t) \\ &+ \frac{1}{2}\int_0^T f_{xx}(t, x(t))d[X, X](t) \\ =& f(0, X(0)) + \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, x(t))\Delta(t)dW(t) \\ &+ \int_0^T f_x(t, (t))\theta(t)dt + \frac{1}{2}\int_0^T f_{xx}(t, x(t))\Delta^2(t)dt \end{aligned}$$

4.Summary of stochastic calculus:

$$\begin{aligned} df(t, x(t)) =& f_t(t, x(t))dt + f_x(t, x(t))dX(t) \\ &+ \frac{1}{2} f_{xx}(t, x(t))dX(t)dX(t) \\ df(t, X(t)) =& f_t(t, X(t))dt + f_x(t, X(t))\Delta(t)dW(t) \\ &+ f_x(t, X(t))\theta(t)dt + \frac{1}{2} f_{xx}(t, X(t))\Delta^2(t)dt \end{aligned}$$

Example 4.4.8 (Generalized geometric Brownian motion).

Define the $It\hat{o}$ process

$$\begin{aligned} X(t) &= \int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds \\ dX(t) &= \sigma(t)dW(t) + \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right)dt \\ dX(t)dX(t) &= \sigma^2(t)dW(t)dW(t) = \sigma^2(t)dt \end{aligned}$$

consider an asset price process

$$S(t) = S(0)e^{x(t)} = s(0)\exp\int_0^t \sigma(s)dw(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds\Big\}$$

$$\begin{aligned} dS(t) &= df(x(t)) \\ &= f'(x(t))dx(t) + \frac{1}{2}f''(x(t))dx(t)dx(t) \end{aligned}$$

Chapter 4 Black-Scholes-Merton-Equation

1. Black - Scholes - Merton Equation

$X(t)$: an agent has a portfolio at each time t
geometric Brownian motion: $dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$ of growth
at each time t, the investor holds $\Delta(t)$ shares of stock. The remainder of the portfolio value, $X(t) - \Delta(t)S(t)$. is invested in the money market account.

$$
\begin{aligned}
&dX(t)\begin{cases}\Delta(t)dS(t) & \text{stock position}\\ r(X(t)-\Delta(t)S(t))dt & \text{cash position}\end{cases}\\
&dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt\\
&\qquad = \Delta(t)(\alpha S(t)dt + \sigma S(t)dW(t)) + r(X(t) - \Delta(t)S(t))dt\\
&\qquad = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)(4.5.2)
\end{aligned}
$$

Understanding of three terms;

(i) an average underlying rate of return r on the portfolio,

$$rX(t)dt$$

ii) a risk premium $\alpha - r$ for investing in the stock.

$$\Delta(t)(\alpha - r)S(t)dt$$

(iii) a volatility term propor vional to the size of the stock investment,

$$\Delta(t)\sigma S(t)dW(t)$$

discrete - time:

$$
\begin{aligned}
&X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n)\\
&X_{n+1} - X_n = \Delta_n(S_{n+1} - S_n) + r(X_n - \Delta_n S_n) \quad (4.5.3)
\end{aligned}
$$

It is analogous to the first line of $(4.5.2)$, except in

$(4.5.3)$ time steps forward one-unit at a time.

consider the discounted stock price: $e^{-rt}S(t)$

According to the Ito-Doebin formula with $f(t,x) = e^{-rt}x$ $d\left(e^{-rt}S(t)\right)$ (discounted stook pria)

$= df(t, S(t))$

$= f_t(t, s(t))dt + f_x(t, s(t))dS(t) + \frac{1}{2}f_{xx}(t, S(t))dS(t)dS(t)$

$= -re^{-rt}S(t)dt + e^{-rt}dS(t)$

$= (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t)$

$d\left(e^{-rt}X(t)\right) \quad (4.5.5)$

$= df(t, X(t))$

$= f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2}f_{xx}(t, X(t))dX(t)dX(t)$

$= -re^{-rt}X(t)dt + e^{-rt}dX(t)$

$= \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) = \Delta(t)d(e^{-r}S(t)) \rightarrow$ the change in discounted portfolio Value is solely due to change in discounted stock price.

2. Evolution of Option Value

$c(t,x)$: denote the value of the call at time t if the stock price at that time is $S(t) = x$

$$\begin{aligned}
&dc(t,s(t)) \\
&= C_t(t,s(t))dt + C_x(t,S(t))dS(t) + \frac{1}{2}C_{xx}(t,s(t))dS(t)dS(t) \\
&= C_t(t,s(t))dt + C_x(t,S(t))C\alpha S(t)dt + \sigma s(t)dw(t)) \\
&+ \frac{1}{2}C_{xx}(t,s(t))\delta^2 s^2(t)dt \\
&= \left[C_t(t,S(t)) + \alpha S(t)C_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)C_{xx}(t,S(t))\right]dt \\
&+ \sigma S(t)C_x(t,S(t))dW(t)
\end{aligned}$$

compute the differential of the discounted option price $e^{-rt}c(t,S(t))$

Let $f(t,x) = e^{-rt}x \quad Ito' -$ Doeblin formula

$$\begin{aligned}
&d\big(e^{-rt}c(t,S(t))\big)(4,5.7) \\
=&df(t,c(t,S(t)) \\
=&f_t(t,c(t,S(t))dt + f_x(t,c(t,S(t))dc(t,S(t))
\end{aligned}$$

$$\begin{aligned}
+&\frac{1}{2}f_{xx}(t,c(t,S(t)))dc(t,S(t))dc(t,S(t)) \\
=&-re^{-rt}c(t,S(t))dt + e^{-rt}dc(t,S(t)) \\
=&e^{-rt}[-rc(t,S(t)) + C_t(t,S(t)) + \alpha S(t)C_x(t,S(t)) \\
&+\frac{1}{2}\sigma^2 s^2(t)C_{xx}(t,S(t))\Big]dt + e^{-rt}\sigma s(t)C_x(t,s(t))dW(t)\Big)
\end{aligned}$$

3. Equating the Evolutions

A short option hedging portfolio stares with some initial capital $X(0) \quad X(t)$ at each time $t \in [0,T]$ agrees with $c(t,S(t))$

$$\Leftrightarrow e^{-rt}X(t) = e^{-rt}c(t,S(t))$$

comparing 4.5 .5 and 4.5 .7

$$\begin{aligned}
&\Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) \\
=&[-rc(t,S(t)) + C_t(t,S(t)) + \alpha S(t)C_x(t,s(t)) \\
&+\frac{1}{2}\sigma^2 S^2(t)C_x(t,s(t))\Big]dt + \sigma S(t)C_x(t,s(t))dW(t)
\end{aligned}$$

First equate $dW(t)$ terms

$$\Delta(t) = C_x(t,S(t)) \quad t \in [0,T)$$

delta - hedging rule

next equate dW(t) terms

$$
\begin{gathered}
(\alpha - r)\sigma(t)C_x(t, s(t)) \\
= -rc(t, s(t)) + C_t(t, s(t)) + \alpha s(t)C_x(t, s(t)) + \frac{1}{2}\sigma^2 S^2(t)C_x(t, s(t)) \\
rc(t, s(t)) = C_t(t, s(t)) + rS(t)C_x(t, s(t)) + \frac{1}{2}\sigma^2 s^2(t)C_{xx}(t, S(t)) \\
t \in [0, T)
\end{gathered}
$$

We should seek a continuous function $C(t, x)$ that is a solution to the Black - Scholes - Merton pde

$$
C_t(t, x) + rxC_x(t, x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t, x) = rc(t, x) \\
t \in [0, T)x \geqslant 0
$$

terminal condition

$$
c(T, x) = (x - K)^+
$$

boundary cindivion: $x = 0$ and $x = \infty$

$$
\begin{aligned}
&c(t, 0) = rc(t, 0) \\
&\text{sol} : c(t, 0) = e^{rt}c(0, 0) \longleftarrow \text{ ODE} \\
&c(t, 0) = 0 \text{ for all } t \in [0, T]
\end{aligned}
$$

As $x \to \infty$, the function $c(t, x)$ grows without bound.

$$
\lim_{x \to \infty} \left[c(t, x) - \left(x - e^{-r(T-t)}k\right)\right] = 0 \text{ for all } t \in [0, T]
$$

The solution to the Black - Scholes - Merton equation with terminal condition : $c(T, x) = (x - k)^+$ and boundary conditions $x = 0$ and $x = \infty$

$$
\begin{aligned}
C(t, x) = &xN(d + (T - t, x)) - Ke^{-r(T-t)}N(d - (T - t, x)) \\
&0 \leq t < T, x > 0 \\
d \pm (r, x) = &\frac{1}{\sigma\sqrt{r}}\left[\log\frac{x}{k} + \left(r \pm \frac{\sigma^2}{2}\right)T\right]
\end{aligned}
$$

N is the cumulative standard normal distribution

$$
\begin{aligned}
&N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{Z^2}{2}}dZ = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{Z^2}{2}}dZ \\
&B\,\text{ss}(T, x, K, r, \sigma) = xN(d + (T, x)) \\
&\text{Black-Scholes-Merton } - Ke^{-rT}N(d_-(T, x)) \\
&\text{function}
\end{aligned}
$$

T : time to expiration

x : current stock price

K, r, σ: strike price, interest rate, stock volatility

4. Put - Call Parity

forward contract: obligates its holder to buy one share of the stock at expiration time T in exchange for payment K.

At expiration, the value of the forward contract is $S(T) - K$.

$f(t, x)$, the value of the forward contract at earlier times $t \in [0, T]$ if the stock price at time t is $S(t) = x$

value of a forward contract: $f(t, x) = x - e^{-r(T-t)}K$
$t = 0: \quad f(t, S(0)) = S(0) - e^{-rT}K$

The forward price of a stock at time t is defined to be the value of K that causes the forward contract at time t to have value 0 . (i.e. $S(t) - e^{-r(T-t)}K = 0$
For $(t) = e^{r(T-t)}S(t)$

European put

pay off $(K - S(T))^+$at T

$$
\begin{aligned}
x - K &= (x-K)^+ - (K-x)^+ \\
f(T, S(T)) &= c(T, S(T)) - p(T, s(T))
\end{aligned}
$$

these values must agree at all previous times $f(t, x) = c(t, x) - p(t, x) \quad x \geqslant 0, 0 \leq t < T$

put - call parity

assumption:
(1) a constant interest rate

(2) the stock is a geometric Brownian motion

(3) constant volatility $\sigma > 0$.

Black - Scholes - Merton put formula

$$
\begin{aligned}
P(t, x) =& x(N(d_+(T-t, x)) - 1) \\
&- Ke^{-r(T-t)}(N(d_-(T-t, x)) - 1) \\
=& Ke^{-r(T-t)}N(-d_-(T-t, x)) - xN(-d_+(T-t, x))
\end{aligned}
$$

chapter 5 Multivariable Stochastic Calculus

1.Multiple Brownian Motions

def: A d-dimensional Brownian motion is a process

$$W(t) = (W_1(t), \cdots W_d(t))$$

with the following properties:

(i) Each $W_i(t)$ is a one dimensional Brownian motion.

(ii) If $i \neq j$, then the process is $W_i(t)$ and $W_j(t)$ are independent.

$\mathcal{F}(t) \leftrightarrow$ associated with a d-dimensional Brownian notion

(iii) (Information accumulates) For $0 \leq s < t$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$

(iv) (Adaptivity) For each $t \geqslant 0$, the random vector $W(t)$ is $\mathcal{F}(t)$ - Measurable.

(v) (Independence of future increments) For $0 \leq t < u$, the vector of increments $W(u) - W(t)$ is independent of $\mathcal{F}(t)$

Each component W_i is a one-dimensional Brownian motion, we have the quadratic variation formula $[W_i, W_i](t) = t$

$$\Rightarrow dW_i(t)dW_i(t) = dt$$

when $i \neq j$. since W_i and W_j are independent,

we have $[W_i, W_j](t) = 0$

$$dW_i(t)dW_j(t) = 0 \quad i \neq j$$

Let $\pi = \{t_0. \cdots, t_n\}$. be a partition of $[0, T]$. For $i \neq j$, sampled cross variation of w_i and w_j on $[0, T]$ to be

$$C_\pi = \sum_{k=0}^{n-1} [W_i(t_{k+1}) - W_i(t_k)][W_j(t_{k+1}) - W_j(t_k)]$$

The increments appearing on the RHos are all independent of one another and all have mean $0, EC_\pi = 0$

$$C_\pi^2 = \sum_{k=0}^{n-1} [W_i(t_{k+1}) - W_i(t_k)]^2 [W_j(t_{k+1}) - W_j(t_k)]^2$$
$$+ 2 \sum_{k+1}^{n-1} [W_i(t_{k+1}) - W_i(t)][W_j(t_{k-1}) - W_j(t_1)] \cdot [W_i(t_{k+1}) - W_i(t_k)]$$
$$[W_j(t_{k+1}) - W_j(t_k)]$$

$$\operatorname{Var}(C_\pi) = EC_\pi^2 = E \sum_{k=0}^{n-1} [W_i(t_{k+1}) - W_i(t_k)]^2 [W_j(t_{k+1}) - W_j(t_k)]^2$$

independent. each has expectation $(t_{k+1}-t_k)$

$$\operatorname{Var}(C\pi)=\sum_{k=0}^{h-1}(t_{k+1}-t_k)^2\le\|\pi\|\cdot\sum_{k=0}^{h-1}(t_{k+1}-t_k)=\|\pi\|\cdot T$$
$$\|\pi\|\to 0\operatorname{Var}(G)\to 0$$

2. ***Itô*-Doeblin Formula for Multiple Processes**

$X(t), Y(t)$ be *Ito* to processes

$$X(t)=X(0)+\int_0^t\theta_1(u)du+\int_0^t\sigma_{11}(u)dW_1(u)+\int_0^t\sigma_{12}(u)dW_2(u)$$
$$Y(t)=Y(0)+\int_0^t\theta_2(u)du+\int_0^t\sigma_{21}(u)dW_1(u)+\int_0^t\sigma_{22}(u)dW_2(u)$$
$$\theta_i(u)\cdot\sigma_{ij}(u):\text{adapted processes}$$

$$dX(t)=\theta_1(t)dt+\sigma_{11}(t)dW_1(t)+\sigma_{12}(t)dW_2(t)$$
$$dY(t)=\theta_2(t)dt+\sigma_{21}(t)dW_1(t)+\sigma_{22}(t)dW_2(t)$$
$$\int_0^t\sigma_{11}(u)dW_1(u)\text{ accumulates quadratic variation at rate}$$
$$\sigma_{11}^2(t)\text{ per unit time,}$$
$$\int_0^t\sigma_{12}(u)dW_2(u)$$
$$\sigma_{12}^2$$

the process $X(t)$ accumulates quadratic variation $\sigma_{11}^2(t)+\sigma_{12}^2(t)$ per unit time.

multiplication rules:

$$dtdt=0\quad dtdW_i(t)=0\quad dW_i(t)dW_i(t)=dt$$
$$dW_i(t)dW_j(t)=0\text{ for }i\neq j$$
$$dY(t)dY(t)=\left(\sigma_{21}^2(t)+\sigma_{22}^2(t)\right)dt$$
$$dX(t)dY(t)=(\sigma_{11}(t)\sigma_{21}(t)+\sigma_{12}(t)\sigma_{22}(t))dt$$
$$\text{for every }T\geqslant 0$$
$$[X,Y](T)=\int_0^T(\sigma_{11}(t)\sigma_{21}(t)+\sigma_{12}(t)\sigma_{22}(t))dt$$

$$\sum_{k=0}^{n-1}\left[X(t_{k+1})-X_{(k_k)}\right)[Y(t_{k+1})-Y(t_k)]$$

Thm 4. 6. 2 (Two-dimensional Ito -Doeblin formula)

$$f(t,x,y).\quad X(t)Y(t):\textit{Itô}\text{ processis}$$

The two-dimensional Itô- Doeblin formula in differential form

$$\begin{aligned}df(t,X,Y)=&ftdt+fxdX+f_ydY+\frac{1}{2}f_{xx}dXdX+f_{xy}dXdY\\&+\frac{1}{2}f_{yy}dYdY\end{aligned}$$

Corollary 4.6.3 (Ito-product rule) Let $X(t), Y(t)$ be Ito process. Then

$$d(X(t)Y(t))=X(t)dY(t)+Y(t)dX(t)+dX(t)dY(t)$$

3.Recognizing a Brownian Motion

Thm 4.6.4

Let $M(t)$, $t \geqslant 0$, be a martingale relative to a filtration $\mathcal{F}(t), t \geqslant 0$. Assume $M(0)=0$. $M(t)$ has continuous paths. and $[M \cdot M](t)=t$ for all $t \geq 0$, then $M(t)$ is a Browian motion.

Pf: idea, to prove $M(t)$ is normally distributed $\boldsymbol{Itô - Docblin}$ formula

$$\begin{aligned} df(t, M(t)) = f_t(t, M(t))dt + f_x(t, M(t))d/M(t) \\ + \frac{1}{2} f_{xx}(t, M(t))dt \end{aligned}$$

integrated form

$$\begin{aligned} f(t, M(t)) = f(0, M(0)) + \int_0^t \left[f_t(s, M(s)) + \frac{1}{2} f_{xx}(s, M(s)) \right] ds \\ + \underbrace{\int_0^t f_x(s, M(s))dM(s)}_{t=0 \Rightarrow \text{ vanish}} \end{aligned}$$

$$\begin{aligned} & Ef(t, M(t)) = f(0, M(0)) + E \int_0^t \left[\frac{\left[f_t(s, M(s)) + \frac{1}{2} f_{xx}(s, M(s)) \right]}{\text{become 0.}} \right] ds \\ & fix\ u, \text{ define } f(t, x) = \exp\left\{ ux - \frac{1}{2} u^2 t \right\} \quad f_x(t, x) = uf(t, x). \\ & f_t(t, x) = -\frac{1}{2} n^2 f(t, x) \quad f_{xx}(t, x) = u^2 f(t, x) \\ & f_t(t, x) + \frac{1}{2} f_{xx}(t, x) = 0 \\ & E\left\{ \exp \mid uM(t) - \frac{1}{2} u^2 t \right\} = 1 \\ & Ee^{uM(t)} = e^{\frac{1}{2} u^2 t} \end{aligned}$$

Thm 4.6.5 (two dimensions)

Let $M_1(t)$ and $M_2(t), t \geqslant 0$, be martingales relative to a $\mathcal{F}(t)$, $t \geqslant 0$. Assume $M_i(0)=0$, $Mi(t)$ has continous paths, and $[M_i, M_i](t)=t$ for all $t=0$. If, in addition. $[M_1, M_2](t)=0$ for all $t \geqslant 0$, then $M_1(t)$ and $M_2(t)$ are independent Brownian motions

Ex 4.6.6 (Correlated stock price) Suppose

$$\begin{aligned} \frac{ds_1(t)}{s_1(t)} &= \alpha_1 dt + \sigma_1 dW_1(t) \\ \frac{ds_2(t)}{s_2(t)} &= \alpha_2 dt + \sigma_2 \left[\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t) \right] \end{aligned}$$

$W_1(t)$ $W_2(t)$ are independent Brownian motions

$$\begin{cases} W_3(t) = \rho W_1(t) + \sqrt{1-\rho^2} W_2(t) \\ dW_3(t)dW_3(t) = P^2 dW_1(t)dW_1(t) + 2p\sqrt{1-\rho^2} dW_1(t)dW_2(t) \end{cases}$$

$+ \left(1-\rho^2\right) dW_2(t)dW_2(t)$

$= p^2 dt + \left(1-p^2\right) dt = dt$

$\Rightarrow [W_3, W_3](t) = t$

$W_3(t)$ is a continuous martingale and $W_3(0) = 0$

$\Rightarrow W_3(t)$ is a Brownian motion

$\dfrac{ds_2(t)}{s_2(t)} = \alpha_2 dt + \sigma_2 dw_3(t)$

$d(W_1(t)W_3(t)) = W_1(t)dW_3(t) + W_3(t)dW_1(t) + dW_1(t)dW_3(t)$

$= W_1(t)dW_3(t) + W_3(t)dW_1(t) + \rho dt$

$$W_1(t)W_3(t) = \underbrace{\int_0^t w_1(s)dw_3(t)}_{E()=0} + \frac{\int_0^t w_3(s)dw_1(s)}{E() = 0} + Pt$$

$E[W_1(t)W_3(t)] = P_t$

$Chapert$ 6 The Black - Scholes Model

1.Notions:

risk of a portfolio: variance of the return

eg. bank account - risk free

highly volatile stock - large variance

Option Values, Payoffs and strategies

$C(S, t)$: Current value of the undelying asset, S

$P(S, t)$: current value of a put, S .

E: the excise price

$S > E$ at expiry: exercise the call option

profit: $S - E$

$S < E$: We would make a loss of $E - S$

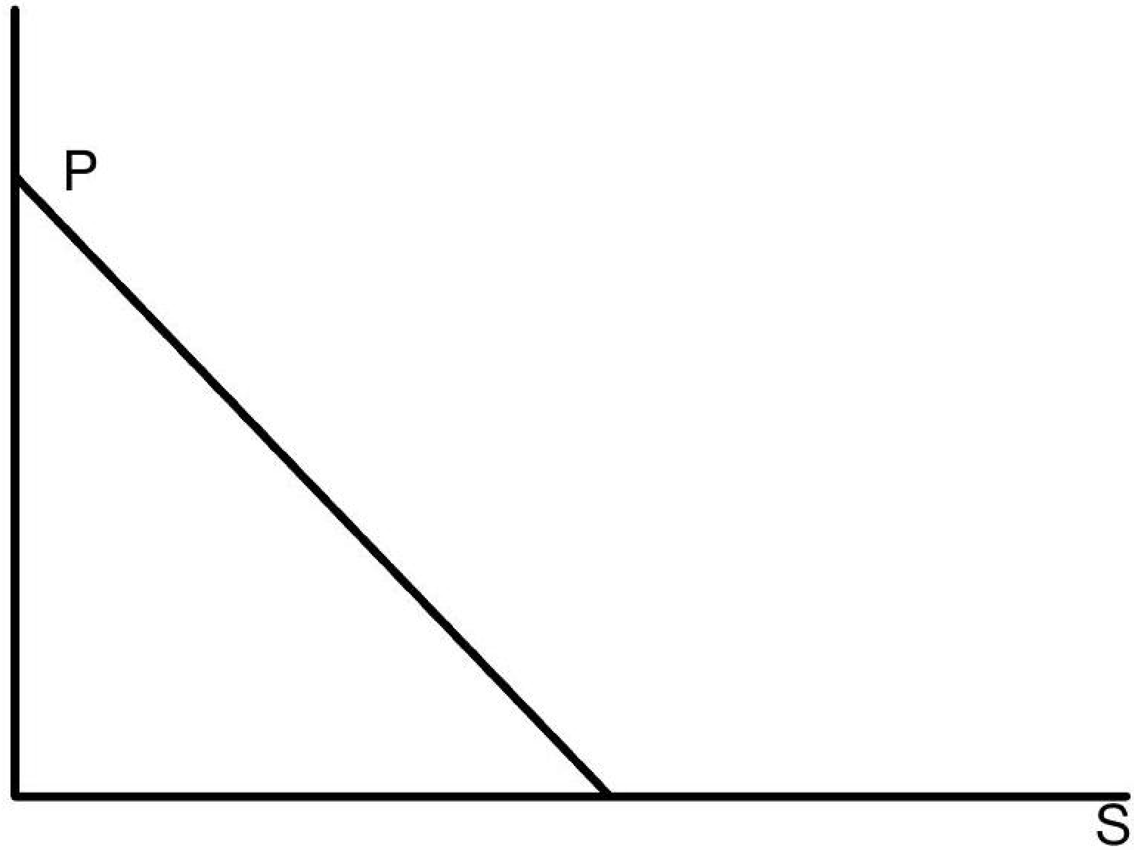
The value of the call option at expiry can be mitten;

$$C(S, T) = \max(S - E, 0)$$

time value: value before expiry

intrinstic value: value at expiry

pay off diagram.



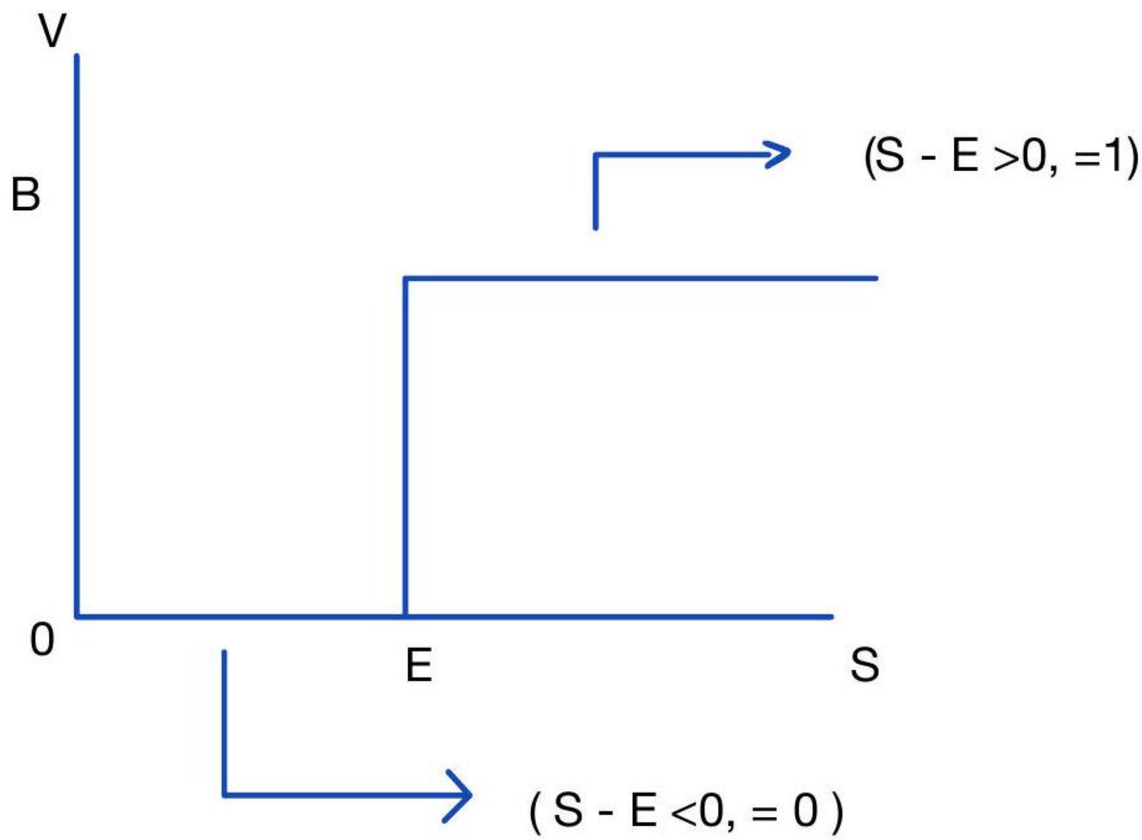
the value of the call option at expiry: $\quad C(S,T) = \max(S-E,0)$

pay off diagram for a par $P(S,t)$.

Example of another pay off

$$B\mathcal{H}(S-E)$$

$\mathcal{H}(\cdot)$: Heaviside function $\begin{cases} 0 & \text{its argument is negative} \\ 1 & \text{otherwise} \end{cases}$



Put - call parity

$$\begin{aligned} S+P-C &= Ee^{-r(T-t)} \\ C-P &= S-Ee^{-r(T-t)} \end{aligned}$$

The Black Scholes Analysis

- The asset price follows the lognorand random walk.
- Short selling is permitted and the assets are divisible We assume that we can buy and sell any number of the underlying asset. and we may sell assets that we do not own,

2.The Black - Scholes Equation

1. Stochastic differential Equations

(1) A stochastic differential equation is an equation of the form

$$\begin{aligned} & dX(u)=\beta(u, X(u))du+\gamma(u, X(u))dW(u) \quad (0,2.1) \\ & \beta(u, x): \text{ drift } \quad \gamma(u, x) \text{ diffusion} \\ & \text{initial condition, } X(t)=x \quad t \geqslant 0 \end{aligned}$$

The problem is then to find a stochastic process $X(T)$, defined for $T \geqslant t$, such that $X(t)=x$

$$X(T)=X(t)+\int_t^T \beta(u, X(u))du+\int_t^T \gamma(u, x(u))dw(u)$$

The sol $X(T)$ at T will be $\mathcal{F}(\tau)$-measurable (ie. $X(\tau)$ only depends on t he path of the Brownian motion up to time T)

Since the initial condition $X(t) = x$ is specified, all that is really needed to determine $X(T)$ is the path of the Brownian motion between t and T.

(2) One dimensional linear stochastic differential equation
$dX(u) = (a(u) + b(u)X(u))du + (y(u) + \delta(u)X(u))dW(u)$ $a(u), b(u), \sigma(u), \gamma(u)$ are nonrandom functions of time

(3) Geometric Brownian Motion

The stochastic differential equation for geometric Brownian motion

$$
\begin{aligned}
&dS(u) = \alpha S(u)du + \delta S(u)dw(u) \\
&\beta(u,x) = \alpha x \quad \gamma(u,x) = \sigma(x) \\
&\text{formula of the sol: initial position: } S(0) \\
&S(t) = s(0)\exp(\sigma W(t) + (\alpha - \frac{1}{2}\sigma^2)ty) \\
&\text{for } T \geqslant t \\
&S(T) = S(0)\exp[\sigma\omega(T) + \left(\alpha - \frac{1}{2}\sigma^2\right)T] \\
&\frac{S(T)}{S(t)} = \exp\sigma(W(T) - W(t)) + \left(\alpha - \frac{1}{2}\sigma^2\right)(T-t)\}
\end{aligned}
$$

If the initial value is $S(t) = x$

$$
S(T) = x\exp\sigma(w(T) - w(t)) + \left(\alpha - \frac{1}{2}\sigma^2\right)(T-t)\}
$$

(4) Hull-White interest rate model

stochastic differential equation

$$
\begin{aligned}
&dR(u) = (a(u) - b(u)R(u))du + \sigma(u)d\overline{w}(u) \\
&\beta(u,r) = a(u) - b(u)r \\
&\gamma(u,r) = \sigma(u)
\end{aligned}
$$

initial condition: $R(t) = r$

first using the stochastic differential equation to compete

$$
\begin{aligned}
d\left(e^{\int_0^u b(v)dv}R(u)\right) &= e^{\int_0^u b(v)dv}(b(u)R(u)du + dR(u)) \\
&= e^{\int_0^u b(v)dv}(\alpha(u)du + \sigma(u)d\tilde{W}(u))
\end{aligned}
$$

integrating both sides from t to T using the initial condition

$$
\begin{aligned}
&R(t) = r \\
&e^{\int_0^u b(v)dv}R(u) = re^{\int_0^t b(v)dv} + \int_t^T e^{\int_0^n b(v)dv}\alpha(u)du \\
&+ \int_t^T e^{\int_0^u b(v)dv}\sigma(u)d\overline{w}(u) \\
&R(T) = re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv}\alpha(u)du \\
&+ \int_t^T e^{-\int_u^T b(v)dv}\sigma(u)d\overline{w}(u)
\end{aligned}
$$

(Recall thm 4.4.9 Itˆo integral of a deterninistic integrand)
Let $W(S), S \geqslant 0$. be a Brownian motion . $\Delta(s)$: non-random function of time,
$I(t)=\int_0^t \Delta(s) dW(s)$.

For each $t \geq 0, I(t)$ is normally distributed with expected value 0 and variance $\int_0^t \Delta^2(s) ds$

(5) The Markov Property

consider stochastic differential equation

$dX(u)=\beta(u, X(u)) du+\gamma(u, X(u)) dW(u)(6.2.1)$.

Let $h(y)$ be a Boral -measurable function.

$$g(t, x)=E^{t, x} h(X(T))$$

$X(T)$ is the sol to 6.2.1 with initial condition $X(t)=x$

we compute $g(t, x)$ numerically by beginning at $X(t)=x$

choose a positive step size δ

than set $X(t+\delta)=x+\beta(t, x) \delta+\gamma(t, x) \sqrt{\delta} \epsilon_1$

Then,

$$\begin{aligned} x(t+2 \delta)= & x(t+\delta)+\beta\left(t+\delta_1 x(t+\delta)\right) \delta \\ & +\gamma(t+\delta, X(t+\delta)) \sqrt{\delta} t_2 \end{aligned}$$

Theorem $(6.3.1)$

Let $X(u), u>0$ be a sol to the stochastic differential equation $(6.2.1)$ with initial condition given at time 0 . Then. for $0 \leq t \leq T$

$$E[h(X(T) \mid \mathcal{F}(t)]=g(t, X(t))$$

Chapter 6 Black - Scholes Analysis

1. Recap of Ito's lemma

with probability, $dX^2 \to dt$ as $dt \to 0$

Taylor Expansion: $df = \frac{df}{ds}ds + \frac{1}{2}\frac{d^2f}{ds^2}ds^2 + \cdots \quad (2.5)$

Stochastic differential equation

$$\frac{ds}{s} = \sigma dx + \mu dt$$

μ : measure of the average rate of growth of the asset price, also known as the drift.

dX sample from a normal distribution

$$\begin{aligned} dS^2 &= (\sigma S dx + \mu S dt)^2 \\ &= \sigma^2 S^2 dx^2 + 2\sigma\mu S^2 dtdx + \mu^2 S^2 dt^2 \\ \text{since } dx &= O(\sqrt{dt}) \end{aligned}$$

To leading order, $ds^2 \to \sigma^2 S^2 dt$

plug into 2.5

$$\begin{aligned} df &= \frac{df}{ds}(\sigma s dx + \mu S dt) + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{ds^2} dt \\ &= \sigma S \frac{df}{ds} dx + \left(\mu S \frac{df}{ds} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{ds^2}\right) dt \quad (2.7) \end{aligned}$$

Expand $f(S + ds, t + dt)$ in a Taylor series about (s, t) to get

$$\begin{aligned} df &= \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial s^2} ds^2 \\ df &= \sigma s \frac{\partial f}{\partial s} dX + \left(\mu s \frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} + \frac{\partial f}{\partial t}\right) dt \end{aligned}$$

2. Analysis of Black Scholes Model

- we have anoption wise value $V(S, T)$

$$\begin{aligned} &dV = \sigma S \frac{\partial V}{\partial S} dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial s^2} + \frac{\partial V}{\partial t}\right) dt \\ &\pi = V - \Delta S \quad (3.4) \\ &d\pi = dV - \Delta dS \\ &d\pi = \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right) dX + \left(\mu s \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial^V}{\partial t} \right. \\ &\left. - \mu \Delta S\right) dt \\ &\text{Choose } \Delta = \frac{\partial V}{\partial S} \quad (3.6) \\ &d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial S^2}\right) dt \end{aligned}$$

No arbitrage

$\Rightarrow$ The return on an amount π invested in riskless assets would see a growth of $r\pi$dt in a time dt

$$r\pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial \sigma^2}\right) dt$$
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial s^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Black - scholes partial differential equation
RMK:
$(1)\Delta = \frac{\partial V}{\partial S}$: rate of change of the value of our option or portfolio of options with respect to S

(2) equation 3.9 does not contain the growth parameter μ, the value of the option is independent of how rapidly/slowly an asset grows

3. Black - Scholes Equation

The most frequent type of partial diff equations in financial problems is the parabolic equation.

specific relationship $V(S,t)$ and its partial derivatives with respect to independent variables S and t.

Typically

$$V(S,t) = V_a(t) \text{ on } s = a$$
$$V(S,t) = V_b(t) \text{ on } s = b$$

$$\text{equation} \begin{cases} \text{backward type: impose a 'final condition'} \\ v(s,t) = V_\tau(s) \\ \text{forward types impose an 'initial' condition on } t = 0 \end{cases}$$

4. Boundary and Final Conditions for European Options

(1) European call:

$C(S,T) = \max(S - E, 0)$ final condition for our PDE (3.10)

Asset - price boundary conditions:

applied at zero asset price. $S = 0 \rightarrow$ pay off is 0

$$C(0,t) = 0 \quad (3,11)$$
$$\text{As } S \rightarrow \infty$$
$$C(S,t) \backsim S \quad (3,12)$$

(2) European put:

$$P(S,T) = \max(E - S, 0)$$

since if S is ever O then it must remain D

$$P(0,t) = Ee^{-r(T-t)}$$

For a time -dependent interest rate we have

$$P(0,t) = Ee^{-\int_t^T r(T)dT}$$

As $S \rightarrow \infty$

$$P(S,t) \to 0$$

5. Black - Scholes Formula for European Options

when r and δ arecinstant, and the exact, explicit sol for the European call is

$$C(s,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

$N(\cdot)$: Cumulative distribution function for a standardized normal random

$$\begin{aligned} N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy \\ d_1 &= \frac{\log\left(\frac{S}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\log(S/E) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

For a put

$$\begin{aligned} &P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1) \\ &\Delta\begin{cases} Call \quad N(d_1) = \frac{\partial C}{\partial S} \\ \text{put } N(d_1) - 1 = \frac{\partial P}{\partial S} \end{cases} \end{aligned}$$

6. PDE

(1) Heat / diffusion equation

$$\begin{gathered} \frac{\partial u}{\partial} = \frac{\partial^2 n}{\partial x^2} \quad (4.1) \\ x \text{ : spatial variable} \qquad \text{T: time variable} \end{gathered}$$

initial value problem

$$\frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial x^2} \quad -\infty < x < \infty$$

with initial data $u(x,0) = u_0(x)$

$$\text{and } u \to 0 \text{ as } x \to \pm\infty$$

Technical point 1: Characteristics of Second Order Linear PDE

$$\begin{aligned} &a(x,\tau)\frac{\partial^2 u}{\partial x^2} + b(x,\tau)\frac{\partial^2 u}{\partial x \partial T} + c(x,\tau)\frac{\partial^2 u}{\partial T^2} + d(x,\tau)\frac{\partial u}{\partial x} \\ &+ e(x,\tau)\frac{\partial u}{\partial T} + f(x,\tau)u + g(x,\tau) = 0 \\ &\text{write } x = x(\xi), T = T(\xi), \text{ then } x(\varepsilon) \text{ and } T(\varepsilon) \text{ satisfy} \\ &a(x,\tau)\left(\frac{d\tau}{d\xi}\right)^2 - b(x,\tau)\frac{dT}{d\xi}\frac{dx}{d\xi} + c(x,\tau)(\frac{dx}{d\xi}\Bigg)^2 = 0 \end{aligned}$$

Parabolic: two real equal roots

diffusion: $b = c = 0$

elliptical: no real characteristics

$Chapter 7$

$PDE\ and\ Variations\ on\ Black\ Scholes\ Model$

1. PDE

(1) The Delta function and the Heaviside function

Eg. I receive money at the rate f(t)dt in a time dt

$$f(t) = \begin{cases} \frac{1}{2t}, & |t| \leq \epsilon \\ 0, & |t| > \epsilon \end{cases}$$

the total payment: $\int_{-\infty}^{\infty} f(t)dt \rightarrow$ equal to 1 independent of ϵ

- delta function, $\delta(t)$: the limit as $\varepsilon \rightarrow 0$

following properties:

I for each ϵ. $\delta_\epsilon(t)$ is piecewise smooth

II. $\int_{-\infty}^{\infty} \delta_\epsilon(t)dt = 1$

III. for each $t \neq 0, \lim_{\epsilon \rightarrow 0} \delta_\varepsilon(t) = 0$

Such a sequence of functions is called a delta sequence.

Ex: $\delta_\epsilon(x) = \frac{1}{2\sqrt{\pi\epsilon}} e^{-\frac{x^2}{4\epsilon}}$

the integral of any member of a delta sequence is well-behaved, being equal to 1.

For any smooth function $\phi(x)$, called a test function:

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \delta_t(x)\phi(x)dx = \phi(0)$$

It is apparent that for any $a, b > 0$

$$\int_{-a}^{b} \delta(x)\phi(x)dx = \phi(0)$$

for any x_0

$$\int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)dx = \phi(x_0)$$

- Heaviside function

$$\mathcal{H}(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geqslant 0 \end{cases}$$

$\mathcal{H}'(x) = \delta(x)$

Ex. M(t) represents the amount of money owned by a person

$$M(t) = \begin{cases} 0 & \text{for } 0 < t < t_0 \\ 0 + D_\delta & \text{for } t \geqslant t_0 \end{cases}$$

MIt) satisfies the differciticl equation

$$\frac{dM}{dt} = D_\delta \delta(t - t_0)$$

(2) initial and boundary conditions

consider what initial and boundary conditions are appropriate for solutions of the diffusion equation.

I. I. V. P. on a finite interval need: whole boundary

$$\begin{aligned}
&\frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial x^2}, -L < x < L, \text{ with } u(x,0) = u_0(x) \\
&u(-L,T) = g-(T), u(L,T) = g_\tau(T) \\
&\frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial x^2}, -L < x < L, \text{ with } u(x,0) = u_0(x) \\
&-\frac{\partial u}{\partial x}(-L,T) = h_(T\Big), \frac{\partial u}{\partial x}(L,T) = h+(T)
\end{aligned}$$

II. I.V.P. on an infinite interval

$$\frac{\partial u}{\partial T} = \frac{\partial^2 n}{\partial x^2}, -\infty < x < \infty, T > 0 \quad \text{with } u(x,0) = u_0(x)$$

2. Variations on BSM

(1) Options or Divident-paying assets

- constant divident yield
  in a time dt the underlying asset pays out a divident Do $D_0 Sdt$
  D_0 is a constant.

divident yield is as defined as the proportion of the asset price paid out per unit time in this way

Use: good model for index options and for short-clated currency options.

consider the effect of the divident payments on the asset pie?

$$dS = SdX + (\mu - D_0)Sdt$$

since we receive $D_0 Sdt$ for every asset held and since ne hold - δ of the underlying, our portfolio changes by an amount. $-D_0 S\Delta dt$

Add it to $d\tau$

$$\begin{aligned}
&d\pi = dV - \Delta ds - D_0 s\Delta t \\
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + (r - D_0)s\frac{\partial V}{\partial s} - rV = 0 \\
&C(s,t) \sim S_e^{-D_0(7-t)} \text{ as } s \to \infty
\end{aligned}$$

the value of a European call option, with dividents

$$
C(S,t)=e^{-D_0(T-t)}SN(d_{10})-Ee^{-r(T-t)}N(d_{20})
$$

$$
\text{where}\begin{aligned}d_{10}&=\frac{\log(S/E)+\left(r-D_0+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\\ d_{20}&=d_1-\sigma\sqrt{T-t}\end{aligned}
$$

(2) Discrete Divident payments

suppose that our asset pays just one dividend during the lifetime of the option, at time t = td.

at time td, holders of the asset receive a payment dy S, where S is the asset price just before the divident is paid

absence of arbitrage

$$
\delta\left(td^+\right)=S(t\bar{d})-dyS(t\bar{d})=S(t\bar{d})(1-dy)
$$

after paying divichent, less valuable

(3) jump conditions for Discrete Dividends.

$t_d d$

$$
\begin{aligned}V\left(S\left(td^-\right),t\vec{d}\right)&=V\left(S\left(t_d^+\right),t_d^+\right)\\ V\left(S,td^-\right)&=V\left(s(1-dy),td^+\right)\end{aligned}
$$

(4) The Call Option with One Dividend Payment

Recall: Black - Scholes equation is backward parabolic.

When a dividend is paid, idea:

- solve BSM back from expiry until just after the dividend date $T>t>td$

ci.e. until $t=t_d^+)$

- Implement the jump condition (6.7) across $t=td$, to find the Values at $t=t_d^-$
- solve B-S equation backwards from $t=t_d^-$, using these values as final data. $t_d>t>0$

$Cd(s,t)$: value of our call option

$C(S,t;E)$: value of a vanilla European call option with exercise price E

$$
\begin{aligned}Cd(S,t)&=C(S,t;E)\text{ for }{t_d}^+\leqslant t\leqslant T\\ Cd(S,t+)&=C_d(S(l-dy),t_{d^+})\\ &=C(S(1-dy),{t_d}^+;E)\end{aligned}
$$

At expirity, this derivative product has value

$$
\begin{aligned}C(S(1-dy),T;E)&=\max(S(1-dy)-E,0)\\ &=(1-dy)\max\left(S-E(1-dy)^{-1},0\right)\end{aligned}
$$

Thus, it is the same as $(1-dy)$ calls with exercise price $E(1-dy)^{-1}$

For times before td, our call has value

$$Cd(S,t) = (1-dy)C\big(S,t;E(1-dy)^{-1}\big)$$

Chapter 9 Exotic Options

Vanilla: European calls and puts

$\rightarrow$ their payffes depend only on the final value of the underlying

Exotic: (path-dependart)

$\rightarrow$ Payoffs depend on the path

1. Maximum of Brownian Motion with Drift

Begin with a Brownian motion $\underline{\tilde{\omega}(t), 0} \leq \tau \leq T$.

zero drift (martingale)

define $\hat{W}(t) = \alpha t + \tilde{w}(t), 0 \leq t \leq T$

$\hat{M}(t) = \max_{0 \leq t \leq T} \hat{W}(t)$

$\hat{W}(0) = 0 \Rightarrow \hat{M}(T) \geqslant 0$

We also have $\hat{W}(T) \leq \hat{M}(T)$

the pair of random variable $(\hat{M}(t), \hat{W}(t))$ tokes values in the set

$\{(m, w); w \leq m, m \geqslant 0\}$

Theorem 7.21 the joins density under $\tilde{P}$ of the pair $(\hat{M}(T), \hat{W}(T))$ is

$$\tilde{f}_{\hat{M}(T), \hat{W}(T)}(m, \omega) = \frac{2(2m-\omega)}{T\sqrt{2\pi T}} e^{\alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2\pi}(2m-\omega)^2}, \omega \leq m, m \geqslant 0$$

Corollary 7.2 .2

$$\tilde{P}(\hat{M}(T) \leq m) = N\left(\frac{m-\alpha T}{\sqrt{T}}\right) - e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right), m \geqslant 0$$

the density under $\tilde{P}$ of the random variable $\hat{M}(T)$ is

$$\tilde{f}_{\hat{M}(T)}(m) = \frac{2}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(m-\alpha T)^2} - 2\alpha e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right), m \geqslant 0$$

and is 0 for $m < 0$

$$\begin{aligned} &\tilde{P}\{\hat{M}(T) \leq m\} \\ &= \int_0^m \int_\omega^m \frac{2(2\mu-\omega)}{T\sqrt{2\pi T}} e^{\alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\mu-\omega)^2} d\mu d\omega \\ &\quad + \int_{-\infty}^0 \int_0^m \frac{2(2\mu-\omega)}{T\sqrt{2\pi T}} e^{\alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\mu-\omega)^2} d\mu d\omega \end{aligned}$$

2. Knock-out Barrier Options

(1) Notations

- up-and-out

If the underlying asset price begins below the barrier and must cross above it to cause the knock out

- down-and-out

has the barrier below the initial asset price and knocks out if the asset price falls below the barrier

- knock in

they pay off zero unless they cross a barrier

We treat an up -and-out call on a geometric Brownian motion.

(2) Up -and -Out Call

our underlying risky asset is geometric Brownian motion

$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t)$

$\tilde{\omega}(t) : 0 \leq t \leq T$, is a Brownian motion under the risk-neutral measure $\tilde{P}$

The solution to the stochastic differential equation for the asset price is

$$S(t) = S(0)e^{\sigma\omega(t)+\left(r-\frac{1}{2}\sigma^2\right)t} = S(0)e^{\sigma\hat{\omega}(t)}$$
$$\hat{W}(t) = \alpha t + \tilde{\omega}(t), \text{ and } \alpha = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$$
$$\text{we define } \hat{M}(T) = \max_{0 \leq tST} \hat{W}(t), \text{ so}$$
$$\max_{0 \leq t \leq T} S(t) = S(0)e^{\sigma\hat{M}(T)}$$

The option knocks out iff $S(0)e^{\sigma\hat{W}(T)} > B$; if $S(0)e^{\sigma\hat{M}(T)} \leq B$, the option pays off

$$(S(T) - K)^+ = \left(S(0)e^{\sigma\hat{W}(T)} - K\right)^+$$

In other words, the payoff of the option is

$$V(T) = \left(S(0)e^{\sigma\hat{\omega}(T)} - k\right)^+ \mathbb{I}_{\{(S(0)}e^{\sigma\hat{\mu}(T)} \leq B\Big\}$$
$$= \left(S(0)e^{\sigma\hat{\omega}(T)} - k\right)\mathbb{I}_{\left\{(0)e^{\sigma\hat{\omega}(T)} \geqslant k, S(0)e^{\sigma\hat{M}(T)} \leq B\right\}}.$$
$$= \left(S(0)e^{\sigma\hat{\omega}(T)} - k\right)\mathbb{I}_{\{\hat{\omega}(T) \geqslant k, \hat{M}(T) \leq b\}}$$
$$\text{where } k = \frac{1}{\sigma}\log\frac{k}{S(0)}, \quad b = \frac{1}{\sigma}\log\frac{B}{S(0)}$$

(3) Black-Scholes-Merton Equation

Theorem

Let $v(t, x)$ denote the price at time t of the up-and-out call under the assumption that the call has not knocked out prior to time t and $S(t) = x$. Then, $v(t, x)$ satisfies the BSM pde

$$v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = rv(t,x)$$

in the rectangle $\{(t,x) : 0 \le t \le T, 0 \le x \le B\}$ and satisfies the boundary conditions

$$\begin{aligned} &v(t,0) \le 0 \quad 0 \le t \le T \\ &v(t,B) = 0 \quad 0 \le t \le T \\ &v(T,x) = (x-k)^+ \quad 0 \le x \le 1 \end{aligned}$$

Real: $v(t,S(t))$ is the value of the option whee the assumption that it has not knocked out prior to t , whereas $V(t)$ is the value of the option without any assumprion.

In particular, if the underlying asset price rises above the barrier B and then returns below the barrier by time t, then $V(t)$ will be 0 because the option has knocked out.

define: $\rho \to$ the first time t at which the asset price reaches the barrier B.

$$\begin{aligned} &S(t) < B \text{ for } 0 \le t \le \rho \\ &S(\rho) = B \\ &\text{the process } e^{-r(t\wedge\rho)} V(t\wedge\rho) = \begin{cases} e^{-rt}V(t) \text{ if } 0 \le t \le \rho \\ e^{-r\rho}V(\rho) \text{ if } \rho < t \le T \end{cases} \end{aligned}$$

is a $\tilde{P}$ martingale

Before t gets to ρ, this is just the martingale $e^{-rt}v(t)$. Once t gets to ρ. although the time parameter t can march on, the value of the process is frozen at $e^{-r\rho}V(\rho)$

Lemma. We have

$$V(t) = v(t, s(t)), \quad 0 \le t \le \rho$$

In particular, $e^{-rt}v(t,s(t))$ is a $\tilde{P}$-martingale up to time ρ, or,

$$e^{-r(t\wedge\rho)} V(t\wedge\rho, S(t\wedge\rho)), 0 \le t \le T \text{ is a martingale under } \tilde{P}$$

(4) Computation of the Price of the up-and-out call

The risk-nentral price at time 0 of the up-and-out call with payoff $V(T)$ given by (7.3.2) is $V(0) = \tilde{E}\left[e^{-rT}V(T)\right]$

we integrate over the region: $\{(m,n); k \le w \le b, w^+ \le m \le b\}$.

when $0 < S(0) \le B$, the time-zero value of the up-and -out call is

$$
\begin{aligned}
V(0) &= \int_k^b \int_{w^+}^b e^{-rT}(S(0)e^{\sigma\omega} - k)\frac{2(2m-\omega)}{T\sqrt{2T\tau T}} e^{\alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-\omega)^2} dm d\omega \\
&= -\int_k^b e^{-rT}(S(0)e^{\sigma\omega} - k)\frac{1}{\sqrt{2\pi T}} e^{\alpha r - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-\omega)^2}\Big|_{m=\omega^+}^{m=b} dr\sigma \\
&= \frac{1}{\sqrt{2\pi T}}\int_k^b (S(0)e^{\sigma\omega} - k)e^{-rT + \alpha\omega^2 - \frac{1}{2}\alpha^2 T - \frac{1}{2T}\omega^2} d\omega \\
&- \frac{1}{\sqrt{2\pi T}}\int_k^b (s(0)e^{\sigma\omega} - k)e^{-rT + \alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2b-w)^2} d\omega \\
&= S(0)I_1 - KI_2 - S(0)I_3 + kI4 \\
&\text{Set } \delta \pm (T, S) = \frac{1}{\sigma\sqrt{T}}\left[\log_S + \left(r \pm \frac{1}{2}\sigma^2\right)T\right] \\
&I_1 = N\left(\delta + \left(T, \frac{S(0)}{K}\right)\right) - N\left(d_-\left(T, \frac{S(0)}{B}\right)\right) \\
&I_2 = e^{-rT}\left[H\left(\delta - \left(T, \frac{S(0)}{K}\right)\right) - N\left(d - \left(T, \frac{S(0)}{B}\right)\right)\right] \\
&I_3 = \left(\frac{S(0)}{B}\right)^{-\frac{2\tau}{\sigma^2} - 1}\left[N\left(\delta_+\left(T, \frac{B^2}{KS(0)}\right)\right) - N\left(\delta + \left(T, \frac{B}{S(0)}\right)\right)\right] \\
&I_4 = e^{-rT}\left(\frac{S(0)}{B}\right)^{-\frac{2T}{B^2} + 1}\left[N\left(\delta_-\left(T, \frac{B^2}{kS(0)}\right)\right) - N\left(\delta_-\left(T, \frac{B}{S(0)}\right)\right)\right]
\end{aligned}
$$

Now let $t \in [0, T)$, and assume the underlying asset price cat time r is $S(t) = X$. Besides, $0 < x \leq B$

If the call has not knocked out prior to time t, its price at time t is obtained by replacing T by the time to expiration $\tau = T - t$ and replacing $S(0)$ by x.

$$
\begin{aligned}
V(t, x) = &x\left[N\left(\delta_+\left(T, \frac{x}{k}\right)\right) - N\left(\delta_+\left(T, \frac{x}{B}\right)\right)\right] \\
&- e^{-rT}k\left[N\left(\delta - \left(T, \frac{x}{k}\right)\right) - N\left(\delta_-\left(T, \frac{x}{B}\right)\right)\right] \\
&- B\left(\frac{x}{B}\right)^{-\frac{2\tau}{\sigma^2}}\left[N\left(\delta + \left(T, \frac{B^2}{kx}\right)\right) - N\left(\delta + \left(T, \frac{B}{x}\right)\right)\right] \\
&+ e^{-rT}k\left(\frac{x}{B}\right)^{-\frac{2T}{\sigma^2} + 1}\left[N\left(\delta - \left(T, \frac{B^2}{kx}\right)\right) - N\left(\delta - \left(T, \frac{B}{x}\right)\right)\right] \\
&0 \leq t < T, 0 < x \leq B.
\end{aligned}
$$

- For $0 \leq t \leq T$, and $x > B$, we hove $v(t, x) = 0$

(because the option knocks out when the asset price exceeds the barrier B.)

We also have $v(t, 0) = 0$ because geometric Brownian motion starting at 0 stays at 0 .