Mathematics

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Pricing Barrier Option Using Branching Process

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Abstract

This study investigates the valuation of barrier options under a scenario where the underlying asset's price follows a branching process in a random environment (BPRE). We develop a mathematical expression for pricing an up-and-out call option, which is a type of barrier option.

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Contents

1 Introduction

Barrier options, also known as knock-in or knock-out options, have gained popularity since their introduction in the over-the-counter market in 1967 due to their flexibility in hedging strategies and cost-effectiveness compared to standard options. These options, including down-and-out, down-and-in, up-and-out, and up-and-in options, are path-dependent. While closed-form solutions for European barrier options exist, handling American or more complex options often requires numerical methods.

Here's a brief explanation of the different types:

- Up-and-Out Option: In an up-and-out option, the option becomes worthless (knocks out) if the price of the underlying asset rises above a predetermined barrier level at any time during the option's lifetime.

- Up-and-In Option: Conversely, in an up-and-in option, the option only becomes active (knocks in) if the price of the underlying asset rises above the barrier level during the option's lifetime.

- Down-and-Out Option: In a down-and-out option, the option becomes worthless if the price of the underlying asset falls below a predetermined barrier level at any time during the option's lifetime.

- Down-and-In Option: In a down-and-in option, the option only becomes active if the price of the underlying asset falls below the barrier level during the option's lifetime.

Most literature on barrier options assumes geometric Brownian motion for the underlying asset, which has limitations as it does not fully capture observed stock market behavior, such as jumps and leptokurtic distributions. Different stochastic processes have been proposed to address these drawbacks, like jump diffusion, constant elasticity of variance (CEV), Lévy stochastic volatility models, and variance gamma (VG) models.

This paper introduces a model using a branching process in a random environment (BPRE) to capture stock price movement, first proposed by Epps in 1996. It devThis model accounts for thick-tailed return distributions, decreasing variance with stock price level, possible price jumps, and bankruptcy. Previous research has shown promising results in pricing European put options and options on individual U.S. equities using the BPRE model, particularly in eliminating the smile effect observed in option prices.

The paper focuses on deriving a formula for pricing European up-and-out call options based on the BPRE model and comparing results with the lognormal model. It provides a mathematical definition of the BPRE model, its properties, and advantages, along with a formula for the equivalent martingale measure (EMM) parameters.

2 Literature Review

Barrier options were initially introduced in the literature by Snyder in 1965[1]. The first documented valuation of a barrier option occurred in Merton's seminal paper [2], focusing on a down-and-out call option. The framework for pricing path-dependent claims, including barrier options, was further developed in Bergman [3].

In recent years, Barrier options have gained popularity due to their affordability compared to standard options, making them useful in various risk management strategies. Valuing single-barrier options is relatively straightforward.

There exists a variety of methods in pricing barrier options.

To evaluate a protected barrier option, the paper [4] adopted the standard Black-Scholes (1973) model [5], assuming frictionless markets, absence of arbitrage, a constant risk-free rate r, a constant dividend yield q, and that the underlying stock price follows geometric Brownian motion with a constant volatility rate $\sigma > 0$. The up-and-out call option we are extending is strick at K and has a barrier B above the initial stock price S_0. Let T denote the first time after t_1 that the stock price hits the barrier, given that the stock price is below the barrier at time t_1. If the stock price never hits the barrier, we set $T = \infty$. The indicator function of set A is denoted by $1(A)$. Using risk-neutral valuation, the value of a European up-and-out call option with a protection period is expressed as:

$$U_0 = E\left\{e^{-r(T-t_1)}R_1(S_1 \geq B) + e^{-rT}1(T < t_2, S_1 < B) + e^{-rt_2}\max\left[0, S_2 - K\right]1(T \geq t_2, S_1 < B)\right\}$$

A methodology similar to that used for Asian options, rooted in Brownian motion properties, facilitating the derivation of a simple expression for the Laplace transform of double-barrier prices [6]. One new method related to pricing barrier options is using forward deep learning to solve forward-backward stochastic differential equations (FBSDEs). It extends forward deep BSDE by incorporating additional nodes in the computational graph to monitor barrier conditions and preserve relevant values at barrier breach or maturity. [7].

Another hybrid method for pricing barrier-style options, combines Laplace transformation and finite-difference approaches. This method eliminates time steps, providing a fast and accurate numerical solution, as well as handling complex barrier-style options with various constraints effectively [8].

In this paper, we will focus on adopting branching process a random environment for pricing barrier options. Specifically, we derive an analytical formula for the price of an up-and-out call option, a typical type of barrier option [9].

3 Model

3.1 Branching Process

Based on the paper [9], we derive the following model in pricing barrier option. The Bienayme-Galton-Watson branching process is a stochastic model used to describe the evolution of a population over discrete generations. It starts with an initial population size and each individual in the population has a random number of offspring according to a specified probability distribution. The process continues for successive generations, where each individual's offspring follow the same distribution independently of other individuals.

Definition: Under the stated assumptions, a discrete-time process Z_t is defined as a (Galton-Watson) branching process if $Z_0 = 1$ and the population of the n-th generation, Z_n for $n \geq 1$, is determined by the formula:

$$Z_n = \sum_{j=1}^{Z_{n-1}} Z_{(n-1)j},$$

where $Z_{(n-1)j}$, for $n \geq 1$ and $j \geq 1$, are independent copies of an integer random variable Z. This naturally accounts for the possibility of extinction (a concept to be further explored in subsequent sections) because if $Z_n = 0$ for a specific generation, then $Z_q = 0$ for all $q > n$. The above definition is derived from certain properties associated with the Galton-Watson process [10].

Let us consider a Bienayme-Galton-Watson branching process, Z_n, $n = 0, 1, 2, \ldots$, with a non-random number of ancestors $Z_0 > 0$, and it is a postive constant, and the offspring probability distribution:

$$P(Z_{n+1} = 0|Z_n = 1) = (1-u),$$

given that the current generation (Z_n) has one individual is $(1-u)$

$$P(Z_{n+1} = k|Z_n = 1) = u \cdot p \cdot (1-p)^{k-1}, \quad k = 1, 2, \ldots,$$

where $0 < u < 1$ and $0 < p < 1$.

The probability generating function (p.g.f.) $f(s) = E[s^{Z_1}|Z_0 = 1]$, which represents the expected number of offsprings in the first generation given a single ancestor, can be expressed using the factorial moments of the distribution. Specifically,

$$f(s) = \frac{1 - m(1-s)}{1 + \frac{v}{2m}(1-s)}, \quad s \in [0, 1],$$

where $m = \frac{u}{p}$ is the offspring mean and

$$v = \frac{2}{1-p} \cdot \frac{p}{m}, \quad \sigma^2 = v + m - m^2 = \frac{u}{p(1-p)} + \frac{(1-u)}{p}$$

are the offspring second moment and the offspring variance.

Suppose $m > 1$, indicating that the branching process is supercritical. It is well known that for the p.g.f. $f_n(s) = E[s^{Z_n}|Z_0 = 1]$, the following relation holds:

$$f_n(s) = \frac{1 - m^n(1-s)}{1 + \frac{v}{2m}\frac{1-m^n}{1-m}(1-s)}. \tag{2.1}$$

which is crucial for calculating the probabilities $P(Z_n = k|Z_0 = 1)$, $k = 0, 1, 2, \ldots$, $n = 1, 2, \ldots$

Theorem 3.1. *: Using derivative methods, we can get*

$$P(Z_n = k|Z_0 = I) = \frac{1}{k!}\frac{d^k(f_n(s))^I}{ds^k}\bigg|_{s=0} \tag{2.2}$$

$k = 0, 1, 2, ..., K.$

Proof. of (2.2)

By differentiating $f_n(s)$, we obtain for $k = 1, 2, 3, \ldots$:

$$\frac{k!m^n\left[\frac{v(1-m^n)}{2m(1-m)}\right]^{k-1}}{\left[1 + \frac{v(1-m^n)}{2m(1-m)}(1-s)\right]^{k+1}}$$

From the properties of the p.g.f., it follows that

$$P(Z_n = k|Z_0 = 1) = f_n^{(k)}(0)\frac{1}{k!},$$

therefore we get for $k = 1, 2, \ldots$, plugging into s = 0 and divide $k!$,

$$P(Z_n = k|Z_0 = 1) = \frac{m^n\left[\frac{v(1-m^n)}{2m(1-m)}\right]^{k-1}}{\left(1 + \frac{v(1-m^n)}{2m(1-m)}\right)^{k+1}}$$

Substituting $s = 0$ in $f_n(s) = \frac{1-m^n(1-s)}{1+\frac{v}{2m}\frac{1-m^n}{1-m}(1-s)}$, we get the probability for extinction in the n-th generation:

$$P(Z_n = 0|Z_0 = 1) = 1 - m^n\frac{1}{1 + \frac{v(1-m^n)}{2m(1-m)}}. \tag{2.4}$$

Substituting $k = 1$ and $s = 1$ in (2.2), we get the first moment of the process:

$$E[Z_n|Z_0 = 1] = m^n, \quad n = 0, 1, 2, \ldots. \tag{2.5}$$

Substituting $k = 2$ and $s = 1$ in (2.2), we obtain the second factorial moment:

$$E[Z_n(Z_n - 1)|Z_0 = 1] = v \cdot m^{n-1}\frac{1 - m^n}{1 - m}, \quad n = 0, 1, 2, \ldots. \tag{2.6}$$

Theorem 3.2. *If the process starts with $Z_0 = I > 1$ number of ancestors, it is a sum of I independent and identically distributed branching processes. Therefore,*

$$E[s^{Z_n}|Z_0 = I] = (f_n(s))^I,$$

and we can calculate the probablity of $Z_n = k$ given $Z_0 = I$ for $k = 1, 2, \ldots$

$$P(Z_n = k|Z_0 = I) = \frac{1}{k!}\frac{d^k(f_n(s))^I}{ds^k}\bigg|_{s=0}.$$

□

Instead of using the derivatives, there is a simple iterative procedure. For any $n > 0$, $Z_0 = I > 1$, and $K \geq 0$, the following steps are proposed:

1. $P(Z_n = k|Z_0 = 1) = \frac{m^n\left[\frac{v(1-m^n)}{2m(1-m)}\right]^{k-1}}{\left(1+\frac{v(1-m^n)}{2m(1-m)}\right)^{k+1}}$ for $k > 0$

   $P(Z_n = 0|Z_0 = 1) = 1 - m^n\frac{1}{1+\frac{v(1-m^n)}{2m(1-m)}}$. for $k = 0$

2. For $Z_0 = 2$, calculate the sum of two independent processes, each starting with $Z_0 = 1$

   $$P(Z_n = k|Z_0 = 2) = \sum_{j=0}^{k} P(Z_n = j|Z_0 = 1)P(Z_n = k - j|Z_0 = 1),$$

   for all $k = 0, 1, 2, \ldots, K$.

3. For $Z_0 = I$, $I \geq 3$, the process is the sum of two independent processes one of which starts

with $Z_0 = I - 1$ and the other starts with $Z_0 = 1$ particle. That is,

$$P(Z_n = k|Z_0 = I) = \sum_{j=0}^{k} P(Z_n = j|Z_0 = I-1)P(Z_n = k-j|Z_0 = 1)$$

for all $k = 0, 1, 2, ..., K$.

Note the probabilities $p_{1k}(n) = P(Z_n = k|Z_0 = 1)$, $k = 0, 1, 2, ..., K$, calculated in point (i) are entered in an upper triangular matrix as follows:

$$P(n) = \begin{pmatrix} p_{10}(n) & p_{11}(n) & p_{12}(n) & \cdots & p_{1K}(n) \\ 0 & p_{10}(n) & p_{11}(n) & \cdots & p_{1,K-1}(n) \\ 0 & 0 & p_{10}(n) & \cdots & p_{1,K-2}(n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_{10}(n) \end{pmatrix}$$

It is not difficult to see that the probabilities $p_{Ik}(n) = P(Z_n = k|Z_0 = I)$, $k = 0, 1, 2, ..., K$, are equal to the $(1, k)$th element of the Ith power of matrix $P(n)$.

3.2 Branching Process in random environment as a price process

BRPE model is established on the study [9]. Let's consider a scenario where we have a supercritical Bienayme-Galton-Watson branching process denoted as Z_n, where n=0, 1, 2,. . . , along with an independent Poisson process N(t), where $t \geq 0$ with an intensity $\lambda > 0$. We define a randomly indexed branching process, termed as a BPRE, as $S(t) = Z_{N(t)}$, $t \geq 0$,

In this context, S(t) denotes the value of a single stock share at time t, measured in increments of the smallest price movement (e.g., 1/100). In this framework, stock prices are defined as composed of discrete "price particles." Each "price particle" in a given period generates a random number of new "price particles," which collectively determine the stock price in the subsequent period. Consequently, by permitting a random number of generations to emerge within each period, the BPRE model generates stock prices continuously over time.

The fact that Z_n and $N(t)$ are independent leads to the following expression for the probability generating function (p.g.f.) of the process $S(t) = Z_{N(t)}$, beginning with $Z_0 = S(0) \geq 1$ ancestors.

$\Phi(t, s) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} (f_n(s)) S(0)$.

Because the initial asset price $S(0)$ remains constant, meaning it is measurable with respect to $A_0 = \{\Omega, \emptyset\}$. we will omit the conditioning on S(0) to simplify the notation. By employing the probability generating function, and after some calculations, we obtain the subsequent expressions for the mean and variance of the process $S(t)$, $t \geq 0$,

$M(t) = E[S(t)] = S(0)e^{\lambda t(m-1)}$, and

$$\begin{aligned}\sigma^2(t) &= \mathrm{Var}[S(t)] \\ &= S(0)^2[e^{\lambda t(m^2-1)} - e^{2\lambda t(m-1)}] + \frac{S(0)\sigma^2[e^{\lambda t(m^2-1)} - e^{\lambda t(m-1)}]}{m(m-1)}\end{aligned}$$

These formulas allow us to examine in detail some of the main properties of the average rate of return $R(t) = \frac{S(t)-S(0)}{S(0)}$ over a period $(0, t)$. The first two moments of the return distribution have the following form:

$$\begin{aligned}E[R(t)] &= e^{\lambda t(m-1)} - 1, \\ \mathrm{Var}[R(t)] &= e^{\lambda t(m^2-1)} - e^{2\lambda t(m-1)} + \frac{1}{S(0)} \left(\frac{\sigma^2(e^{\lambda t(m^2-1)} - e^{\lambda t(m-1)})}{m(m-1)} \right).\end{aligned}$$

The positive coefficient of $\frac{1}{S(0)}$ in the variance representation arises due to our examination of a supercritical Bienayme-Galton-Watson branching process, where $m > 1$. Consequently, the variance of the return is inversely proportional to the stock price. This phenomenon, known as the "leverage effect," is inherent in the model.

Formulas for the skewness $\gamma_1[R(t)]$ and kurtosis $\gamma_2[R(t)]$ are more intricate. However, if $m = 1$, the standardized third and fourth moments have simpler expressions:

$$\gamma_1[R(t)] = \frac{E[R(t)^3]}{(\mathrm{Var}[R(t)])^{3/2}} = 3\sigma \left(\frac{\sqrt{\lambda t}}{2S(0)} + \frac{1}{2\sqrt{\lambda t}\sqrt{S(0)}} \right),$$

$$\gamma_2[R(t)] = \frac{E[R(t)^4]}{(E[R(t)^2])^2 - 3} = 3\frac{1}{\lambda t} + \frac{1}{S(0)\lambda t} + \frac{2\sigma^2}{S(0)}(\lambda t + 1 + \frac{1}{\lambda t}).$$

Estimations of m using daily stock data are slightly above one, indicating that the last two expressions closely approximate the higher moments during short trading periods. Skewness is consistently positive, decreasing with $S(0)$, and generally increasing with the return period t. Kurtosis is always positive, suggesting fatter tails compared to the normal distribution. Kurtosis decreases with t for t values less than $3S(0) + 2\sigma^2 + \frac{1}{\lambda\sigma\sqrt{2}}$. This value is generally greater than 22.1, implying that daily returns exhibit fatter tails than weekly and monthly returns. This observation is empirically supported by market data and is termed as aggregational normality.

3.3 Option Pricing Using Branching Process

The discrete nature of $S(t)$ under BPRE dynamics poses challenges in replicating nonlinear payoff structures solely with the underlying asset and riskless bonds. Consequently, there isn't a single Equivalent Martingale Measure under which derivatives can be priced as discounted expected values. It's worth recalling that the process $Z_n m^{-n}$, where $n = 0, 1, 2, \ldots$, is a martingale. Likewise, the process $S(t)$ shares a similar property, facilitating the identification of the EMM needed for option pricing. I derive the following theorem based on [9]

Theorem 2.1 Under the conditions (i) a supercritical ($m > 1$) Bienayme-Galton-Watson branching process Z_n, $n = 0, 1, 2, \ldots$, defined in the previous section and (ii) an independent Poisson process $N(t)$, $t \geq 0$ with intensity $\lambda > 0$. Define the randomly indexed branching process, which is in fact a BPRE, $S(t) = Z_{N(t)}$, $t \geq 0$. The process $S(t)e^{-\lambda t(m-1)}$, where $t \geq 0$, is a nonnegative martingale.

The proof involves demonstrating that for $t \geq 0$ and $\tau \geq 0$:

$$E[e^{-\lambda(t+\tau)(m-1)} S(t+\tau) | e^{-\lambda t(m-1)} S(t)] = e^{-\lambda t(m-1)} S(t)$$

or equivalently:

$$E[S(t+\tau) | S(t)] = e^{\lambda\tau(m-1)} S(t)$$

Since the processes $N(t)$ and Z_n are time-homogeneous, and using the main property of branching processes, we have

$$
\begin{aligned}
E[S(t+\tau)|S(t)] &= E[Z_N(t+\tau)|Z_N(t)] \\
&= E\left[\sum_{i=1}^{Z_N(t)} Z_{iN(t+\tau)-N(t)}|Z_N(t)\right] \\
&= E\left[\sum_{i=1}^{Z_N(t)} Z_{iN(\tau)}|Z_N(t)\right] \\
&= Z_N(t)E[Z_{1N(\tau)}] \\
&= S(t)E[Z_{1N(\tau)}],
\end{aligned}
$$

where $Z_{iN(\tau)}$ are independent and identically distributed branching processes that are independent of $Z_N(t)$, and each of them starts with one ancestor. Using

$$M(t) = E[S(t)] = S(0)e^{\lambda t(m-1)}$$

with $S(0) = 1$, we get $E[Z_N(\tau)|Z_0 = 1] = e^{\lambda\tau(m-1)}$. This completes the proof of

$$E[S(t+\tau)|S(t)] = e^{\lambda\tau(m-1)}S(t)$$

and the theorem. From

$$M(t) = E[S(t)] = S(0)e^{\lambda t(m-1)}$$

, it follows that $S(t)e^{-rt}$, $t \in [0, T]$, has mean

$$E[S(t)e^{-rt}] = e^{(\lambda(m-1)-r)t}S(0).$$

Using Theorem 2.1, we conclude that the discounted stock price process $S(t)e^{-rt}$ is a martingale under certain conditions regarding the parameters governing the distribution of $S(t)$. Specifically, when the condition $\lambda(m-1) = r$ holds true, alternatively expressed as $\lambda u - p = r$ or $\lambda u(1-p) = r$, we derive $u = p(1 + \frac{r}{\lambda})$ (Eq. 2.9). This equation serves as the foundation for defining the Equivalent Martingale Measure (EMM) Y as follows:

1. We establish Y to establish the real-world measure P on the elementary sets of the Poisson process, denoted as $\{N_{t_0} = n_0, N_{t_1} = n_1, ..., N_{t_k} = n_k\}$.
2. On the elementary sets of the branching process, Y is defined such that:

$$Y(Z_{n+1} = 0 | Z_n = 1) = (1 - \hat{u})$$

$$Y(Z_{n+1} = k | Z_n = 1) = \hat{u} p (1 - p)^{k-1}$$

for $k = 1, 2, ...$, where $\hat{u}$ is equal to $p(1 + r/\lambda)$. In scenarios where $p(1 + \frac{r}{\lambda}) \geq 1$, adjustments to the parameter p (with $0 < p < 1$) are used to ensure $\hat{u}$ remains within the interval $(0, 1)$.

The selection of $\hat{u}$ ensures that $0 < \hat{u} < 1$, thereby preserving sets with zero measure under the real measure P; hence, all sets with zero measure under P retain zero measure under Y. Consequently, these two measures are deemed equivalent. Notably, by virtue of its definition, the discounted process $S(t)e^{-rt}$ is a martingale under Y. Henceforth, we shall exclusively operate within the framework of the risk-neutral probability, denoted simply as P. In the numerical illustrations provided in Section 4, we maintain the estimated values of p and λ while adjusting the parameter u to satisfy $u = p(1 + \frac{r}{\lambda})$.

Later on, we'll utilize the following expression, originally derived by Mitov and Mitov [22], to compute the price of a European call option:

$$C(0) = S(0) - e^{-rT} K + e^{-(r+\lambda)T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} K \sum_{k=0}^{K} (K - k) P(Z_n = k | Z_0 = S(0)) \quad (2.10)$$

In this equation,

$$
\begin{aligned}
K &: \text{strike price} \\
T &: \text{time to maturity of the option} \\
r &: \text{risk-free interest rate} \\
S(0) &: \text{current stock price} \\
\lambda &: \text{the intensity of the Poisson process.}
\end{aligned}
$$

For practical applications, an approximation can be used:

$$C(0) \approx S(0) - Ke^{-rT} + e^{-(r+\lambda)T} N \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} K \sum_{k=0}^{K} (K-k) P(Z_n = k | Z_0 = S(0)), \quad (2.11)$$

Here, the value of N can be determined such that the error from the approximation is less than ϵ. The probabilities $P(Z_n = k|Z_0 = S(0))$ for $k = 0, 1, 2, ..., K$ and $n = 1, 2, ..., N$ are computed using the iterative procedure illustrated at the end of Section 2.1.

Barrier Option Pricing

- Knock-in barrier option: This type of option exercises when the underlying asset reaches a specific price threshold. The option holder gains the right to exercise it only after this threshold is met in the market.

  If the price hits the designated level at any point during the option's term, it becomes a standard option and is valued accordingly. However, if the price never reaches this level, the knock-in barrier option expires worthless.

  There are two subtypes: up-and-in and down-and-in. An up-and-in option exercises when the asset's price goes above the predetermined barrier, while a down-and-in option exercises when the asset's price falls below the set barrier.

- Knock-out barrier option:

These options become invalid if the underlying asset reaches a specified barrier during the contract's duration. Knock-out barrier options can be further categorized into up-and-out and down-and-out options. An up-and-out option ceases to exist when the asset's price exceeds the barrier set above its initial price. A down-and-out option ceases to exist when the asset's price falls below the barrier set below its initial price. If the asset reaches the barrier at any point during the option's term, the option is terminated or knocked out.

There are more complex barrier options, but in this paper, we focus only on an up-and-out call option on a BPRE process. The methodology we develop can be also applied to up-and-in call options. For the rest we can use in-out parity2 and the price of the standard call option given in (2.10) and (2.11).

Theorem 3.1. *The price of the barrier option using branching process is given by the approximation*

$$C_{uo}(0) \approx e^{-(r+\lambda)T} \sum_{n=0}^{N} \frac{(\lambda T)^n}{n!} \sum_{j=k}^{B} (j-K) \cdot P(Z_n = j, M_n \leq B | Z_0 = S(0))$$

with an error less than

Proof.

$$C_{uo}(T) \triangleq \begin{cases} \max(0, S(T) - K), & \text{if } M(T) \leq B, \\ 0, & \text{if } M(T) > B, \end{cases}$$

Given $M(T) = \max_{0 \leq t \leq T} S(t)$, applying martingale pricing yields the option's current price:

$$C_{uo}(0) = e^{-rT} \mathbb{E}[C_{uo}(T)].$$

Here, the expectation is under the risk-neutral measure $\mathbb{P}$. Hence,

$$C_{uo}(0) = e^{-rT} \mathbb{E}[\max(0, S(T) - K) \mathbf{1}_{\{M(T) \leq B\}}]$$

$$= e^{-rT} \sum_{i=0}^{B} \max(0, i - K) P(S(T) = i, M(T) \leq B),$$

where $\mathbf{1}_A$ denotes the indicator function of event A. Considering $B > K$, we get

$$C_{uo}(0) = e^{-rT} \sum_{j=K}^{B} (j-K) P(S(T)=j, M(T) \le B).$$

By

$$P(S(T)=k, M(T) \le B) = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} e^{-\lambda T} P(Z_n = k, M_n \le B | Z_0 = S(0)).$$

$$C_{uo}(0) = e^{-(r+\lambda)T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} \sum_{i=K}^{B} (i-K) P(Z_n = i, M(T) \le B | Z_0 = S(0))$$

$$+ e^{-(r+\lambda)T} \sum_{n=N+1}^{\infty} \frac{(\lambda T)^n}{n!} \sum_{i=K}^{B} (i-K) P(Z_n = i, M(T) \le B | Z_0 = S(0)).$$

Since

$$\sum_{i=K}^{B} (i-K) P(Z_n = i, M(T) \le B | Z_0 = S(0)) \le \sum_{i=K}^{B} (i-K) = \frac{(B-K)(B-K+1)}{2},$$

then

$$e^{-(r+\lambda)T} \sum_{n=N+1}^{\infty} \frac{(\lambda T)^n}{n!} \sum_{i=K}^{B} (i-K) P(Z_n = i, M(T) \le B | Z_0 = S(0))$$

$$\le e^{-(r+\lambda)T} \sum_{n=N+1}^{\infty} \frac{(\lambda T)^n}{n!} \frac{(B-K)(B-K+1)}{2}.$$

Infinite series can be computed with an arbitrarily small precision $\epsilon > 0$, hence the number N can be determined such that

$$e^{-(r+\lambda)T} \frac{(B-K)(B-K+1)}{2} \sum_{n=N+1}^{\infty} \frac{(\lambda T)^n}{n!} < \epsilon,$$

given the known values of r, λ, T, B, K, and $S(0)$. Consequently, we can approximate

$$C_{uo}(0) \approx e^{-(r+\lambda)T} \sum_{n=0}^{N} \frac{(\lambda T)^n}{n!} \sum_{j=K}^{B} (j-K) P(Z_n = j, M_n \le B | Z_0 = S(0)), \quad (3.1)$$

with an error less than ϵ. The technique for computing the probabilities $P(Z_n = j, M_n \le B | Z_0 = S(0))$ utilized in equation (3.1) is provided in the subsequent theorem.

Theorem 3.2. *If $\{Z_n, n = 0, 1, 2, \ldots\}$ represents a Bienayme-Galton-Watson branching process with transition probabilities $p_{ij} = P(Z_1 = j | Z_0 = i)$, where $i, j = 0, 1, 2, \ldots$, and B is a positive integer, then the probability*

$$P(Z_n = j, M_n \leq B | Z_0 = i)$$

is determined by the (i, j) element of the n-th power of the matrix $P(B, 1)$. Here, $p_{ij}(B, 1)$ is defined as

$$p_{ij}(B,1) = P(Z_1 = j, M_1 \leq B | Z_0 = i) = \begin{cases} p_{ij}, & 1 \leq i, j \leq B, \\ 1, & i = 0, j = 0, \\ 0, & i = 0, j > 0, \\ 0, & B < i, j. \end{cases}$$

Proof. For the conditional probabilities $P(Z_2 = j, M_2 \leq B | Z_0 = i)$ calculated for the second generation, we can express them as:

$$p_{ij}(B, 2) = P(Z_2 = j, M_2 \leq B | Z_0 = i)$$

$$= \sum_{l=0}^{\infty} P(Z_2 = j, M_2 \leq B | Z_1 = l) P(Z_1 = l, M_1 \leq B | Z_0 = i)$$

$$= \sum_{l=1}^{B} P(Z_2 = j, M_2 \leq B | Z_1 = l) P(Z_1 = l, M_1 \leq B | Z_0 = i)$$

$$= \sum_{l=1}^{B} P(Z_1 = j, M_1 \leq B | Z_0 = l) P(Z_1 = l, M_1 \leq B | Z_0 = i)$$

$$= \sum_{l=1}^{B} p_{il} p_{lj}, \quad 1 \leq i, j \leq B.$$

Therefore, $P(B, 2) = (p_{ij}(B, 2))_{i,j=1,\ldots,B} = P(B, 1)^2$. By repeating this procedure, we can deduce by induction that $P(B, n) = P(B, 1)^n$, which confirms the theorem statement.

□

4 Conclusion

In conclusion, this paper introduces a novel approach to pricing barrier options using a branching process in a random environment (BPRE). The model combines elements of a supercritical Bienayme-Galton-Watson branching process with an independent Poisson process to generate stock prices continuously over time. By establishing the Equivalent Martingale Measure (EMM), the discounted stock price process is shown to be a martingale under certain conditions, facilitating option pricing.

Specifically, the paper focuses on deriving a formula for pricing European up-and-out call options based on the BPRE model. The methodology presented in this paper provides a promising framework for pricing barrier options, offering insights for financial practitioners. Future research directions may explore extensions of the model to different types of barrier options and further empirical validation using market data. Overall, the BPRE model contributes to the ongoing advancement of quantitative finance by offering innovative solutions to pricing and risk management challenges in the derivatives market.

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